hp-version a priori error estimates of a DG-CG method for the linear wave equation

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01

PDE setting

Model problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $f \in L^2(L^2)$,

$$\begin{cases} u'' - \Delta_{\boldsymbol{x}} u = f & \text{in } (0,T] \times \Omega \\ u = 0 & \text{on } (0,T] \times \partial\Omega \\ u(0) = u_0 \in H_0^1(\Omega), \quad u'(0) = u_1 \in L^2(\Omega). \end{cases}$$

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Given $X := H^2(H^{-1}) \cap L^2(H_0^1) \cap H^1(L^2)$, $Y := L^2(H_0^1)$, we consider the weak formulation: Find $u \in X$ such that for all $v \in Y$,

$$\int_0^T [\langle u'', v\rangle + (\nabla_{\boldsymbol{x}} u, \nabla_{\boldsymbol{x}} v)] = \int_0^T (f, v),$$

with $u(0) = u_0$ and $u'(0) = u_1$.
Energy conservation: $E(t) = E(0)$, $\forall t \in (0,T]$,
$E(t) := \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2$.

Hamiltonian formulation:

$$\begin{cases} u' - v = 0, \\ v' - \Delta_{\boldsymbol{x}} u = f, \end{cases} \qquad \text{in } (0,T] \times \Omega.$$

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Friedrichs formulation:

$$\begin{cases} u' + \nabla_{\boldsymbol{x}} \cdot \mathbf{v} = f, \\ \mathbf{v}' + \nabla_{\boldsymbol{x}} u = 0, \end{cases} \qquad \text{in } (0,T] \times \Omega.$$

- G. M. Hulbert and Th. J. R. Hughes, 1990:
  space-time DG, second order form.
- C. Johnson, 1993:
  time DG + space CG, second order form.
- O. Karakashian and Ch. Makridakis, 2005:
  time CG + space DG, Hamiltonian form.
- E. Burman, A. Ern, and M. A. Fernandez, 2010:
  explicit Runge–Kutta + stabilized FEM, Friedrichs form.
- N. J. Walkington, 2014
  time DG-CG + space CG, second-order form.

02

Numerical scheme

- V_h: H^1-conforming finite element space of order $p_{\boldsymbol{x}}$.

Discrete spaces

- V_h: H^1-conforming finite element space of order $p_{\boldsymbol{x}}$.
- $X_{h,\tau} := \{v_{h,\tau} \in C^0(0,T;V_h) \mid v_{h,\tau}|_{I_n} \in \mathbb{P}_{p_n^t}(V_h), v_{h,\tau}(0) = u_{0,h}\}$.

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- $\left[\left[u'_{h,\tau}\right]\right](t_{n-1}) := u'_{h,\tau}(t_{n-1}^+) - u'_{h,\tau}(t_{n-1}^-)$,
  with $u'_{h,\tau}|_{I_1}(t_0^-) := u_{1,h}$.

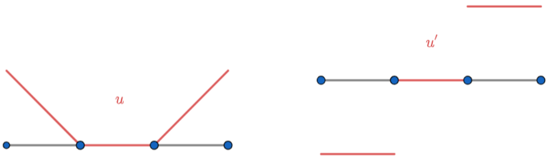

Figure: Illustration of $X_{h,\tau}$

Find $u_{h,\tau} \in X_{h,\tau}$ such that $u_{h,\tau}(t_{n-1}^{+}) := u_{h,\tau}(t_{n-1}^{-})$ and

$$\int_{I_n} \left[(u''_{h,\tau}, v_{h,\tau}) + (\nabla_{\boldsymbol{x}} u_{h,\tau}, \nabla_{\boldsymbol{x}} v_{h,\tau}) \right] + \left(\left[\!\left[u'_{h,\tau} \right]\!\right] (t_{n-1}), v_{h,\tau}(t_{n-1}^{+}) \right)$$
$$= \int_{I_n} (f, v_{h,\tau}), \quad \forall v_{h,\tau} \in \mathbb{P}_{p_n^t - 1}(V_h) \quad \forall n = 1, ..., N.$$

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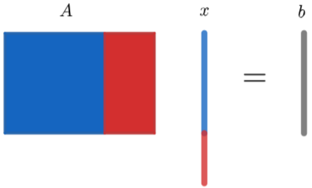

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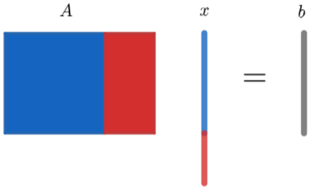

Red unknowns: Determined by the continuity in time.

03

Numerical analysis

- Investigate energy conservation.

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- https://arxiv.org/abs/2411.03264
- https://inria.hal.science/hal-04768144

Continuous level: $E(t) = \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2$,

$$f = 0 \Rightarrow E(t) = E(0), \qquad \forall t \in J.$$

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Discrete level:

$$E(t_n^-) < E(0), \qquad \forall n = 1, ..., N.$$

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How large is the numerical dissipation?

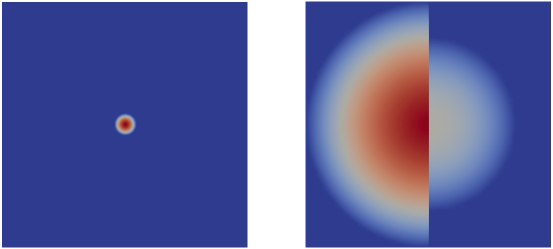

Figure: Initial data. Left: u_0; Right: u_1.

p_n^t	2	3	4	5
$\frac{E(0,U)-E(T,U)}{E(0,U)}$	1.79e-2	1.33e-4	5.48e-7	1.63e-9

$$\int_0^t [(u'', u') + a(u, u')] = \int_0^t (f, u')$$

$$\Rightarrow \sup_{0 \le t \le T} (\|u'(t)\| + \|\nabla u(t)\|) \le C(\|\nabla u_0\| + \|u_1\| + \|f\|_{L^1(L^2)}).$$

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Discrete level:

$$\int_{I_n} [(u''_{h,\tau}, u'_{h,\tau}) + a(u_{h,\tau}, u'_{h,\tau})] + (\llbracket u_{h,\tau} \rrbracket (t_{n-1}), u'_{h,\tau}(t^+_{n-1}))$$
$$= \int_{I_n} (f, u'_{h,\tau})$$
$$\Rightarrow \|u'_{h,\tau}(t_n)\|^2 + \|\nabla u_{h,\tau}(t_n)\|^2$$
$$\le C(\|\nabla u_{0,h}\| + \|u_{1,h}\| + \|f\|_{L^1(L^2)}) \sup_{0 \le t \le t_n} (\|u'_{h,\tau}(t)\|^2 + \|\nabla u_{h,\tau}(t)\|^2)^{\frac{1}{2}}$$

Test the scheme with $(1-\frac{t-t_{n-1}}{4(2p_n^t+1)})u'_{h,\tau} \approx \exp(-\lambda t)u'$ from [N. J. Walkington,2014]

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Theorem (Stability)

Let $u_{h,\tau}$ be the numerical solution, m given by

$$\max_{t \in I_m} E(u_{h,\tau}(t)) = \max_{t \in (0,T]} (E(u_{h,\tau}(t))),$$

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$$\sup_{0<t\leq T} E(u_{h,\tau}(t)) + (p_m^t)^3 \sum_{n=1}^{m} \| [[u'_{h,\tau}]] (t_{n-1})\|^2$$

$$\lesssim (p_m^t)^3 E(u_{h,\tau}(0)) + (p_m^t)^6 \int_0^{t_m} \|f\|^2,$$

Error decomposition

$$e_{h,\tau} := u - u_{h,\tau} = (u - \tilde{u}_{h,\tau}) + (\tilde{u}_{h,\tau} - u_{h,\tau}),$$

with $\tilde{u}_{h,\tau}$ to be specified later.

- $(\nabla_{\boldsymbol{x}}(u - \Pi_h^{\mathcal{E}} u), \nabla_{\boldsymbol{x}} v_h) = 0, \quad \forall v_h \in V_h.$

Control on $u - \tilde{u}_{h,\tau}$

- $(\nabla_{\boldsymbol{x}}(u - \Pi_h^{\mathcal{E}} u), \nabla_{\boldsymbol{x}} v_h) = 0, \quad \forall v_h \in V_h$.
- $\mathcal{P}_{\mathbf{p}^t} : C^1(L^2) \to X_{h,\tau}$ such that for all $q \in \mathbb{P}_{p_n^t - 2}(I_n; L^2)$,

$$\int_{I_n} (u'', q) = \int_{I_n} ((\mathcal{P}_{\mathbf{p}^t} u)'', q) + (\llbracket (\mathcal{P}_{\mathbf{p}^t} u)' \rrbracket (t_{n-1}), q(t_{n-1}^+)),$$

with $\mathcal{P}_{\mathbf{p}^t} u(t_{n-1}) := u(t_{n-1})$, $\mathcal{P}_{\mathbf{p}^t} u(t_n) := u(t_n)$ and $(\mathcal{P}_{\mathbf{p}^t} u)'(t_n^-) := u'(t_n)$.

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- [N. J. Walkington, 2014]: h-version analysis for $\mathcal{P}_{\mathbf{p}^t}$.
- $\tilde{u}_{h,\tau} := \mathcal{P}_{\mathbf{p}^t} \Pi_h^{\mathcal{E}} u$.

Lemma (Control on $\tilde{u}_{h,\tau} - u_{h,\tau}$)

Let $\xi = \tilde{u}_{h,\tau} - u_{h,\tau}$, $m = m(\xi)$. *Assume that* $u \in H^2(I_n; L^2)$ *and* $u'' \in L^2(I_n; L^2)$ *for all* n. *Then,*

$$\|\xi'\|_{L^\infty(I_m;L^2)} + |\xi|^2_{L^\infty(I_m;H^1)} + (p_m^t)^3 \sum_{n=1}^{m} \| \left[\!\left[\xi'\right]\!\right] (t_{n-1})\|^2$$

$$\lesssim (p_m^t)^3 \Big(\|\nabla_{\boldsymbol{x}}(u_0 - u_{0,h})\|^2 + \|u_1 - u_{1,h}\|^2 + \|u_1 - (\mathcal{P}_{\mathbf{p}^t} u)'(0)\|^2 \Big)$$

$$+ (p_m^t)^3 \|(\mathcal{P}_{\mathbf{p}^t}(I - \Pi_h^{\mathcal{E}})u)'(0)\|^2$$

$$+ (p_m^t)^6 \sum_{n=1}^{m} \Big(\|(I - \mathcal{P}_{\mathbf{p}^t})\Delta_{\boldsymbol{x}} u\|^2_{L^2(L^2)} + \|(I - \Pi_h^{\mathcal{E}})u''\|^2_{L^2(L^2)} \Big).$$

Corollary (Uniform refine: $h = O(\tau)$, $p_n^t = p_{\boldsymbol{x}} = p$)

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- *For sufficiently regular solution* u,

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- *For $u^{(k)} \in L^\infty(H^{s-k})$, $\forall k = 0, ..., s$ with $s > 2$,*

$$\|e'_{h,\tau}\|^2_{L^\infty(L^2)} + |e_{h,\tau}|^2_{L^\infty(H^1)} \lesssim \frac{\tau^{2s}}{p^{c(s)}}.$$

04

Numerical experiments

The numerical experiments are conducted with the Gridap.jl library in the Julia programming language.

Test 1

$$u(x,y,t) = (1-x^2)(1-y^2)\cos(4t),\ u \in C^\infty(C^\infty).$$

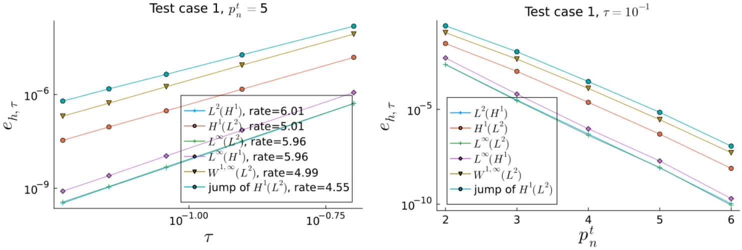
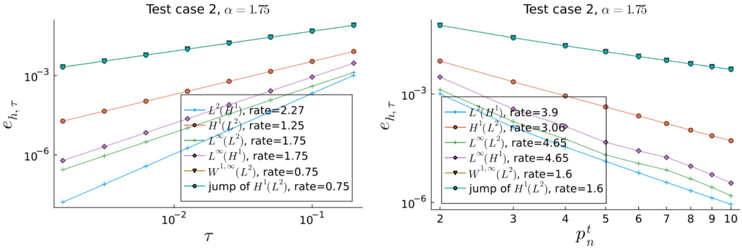


Figure: Left: τ-refinement, right: p_n^t-refinement.

$$u(x,y,t) = (1-x^2)(1-y^2)t^\alpha,\ u \in H^{2.25}(C^\infty).$$



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Thank you!