EASTERN INSTITUTE OF TECHNOLOGY, NINGBO

A posteriori error analysis and adaptivity of a space-time method for the linear wave equation

CMAM, MS17, 2026

Z. Dong[1], E. H. Georgoulis[2], L. Mascotto[3], Z. Wang[4]

Inria Centre de Paris & ENPC, Institut Polytechnique de Paris[1]
U. Heriot-Watt & N.T.U. Athens & FORTH[2]
U. Milano-Bicocca & IMATI & U. Vienna[3]
Eastern Institute of Technology, Ningbo[4]

1. Model problem
2. Literature review
3. Numerical scheme & main result
4. Proof details
5. Numerical tests

Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$ be a bounded open set, $f \in L^2(L^2)$,

$$\begin{cases} u'' - \Delta u = f & \text{in } (0,T] \times \Omega \\ u = 0 & \text{on } (0,T] \times \partial\Omega \\ u(0) = u_0 \in H_0^1(\Omega), \quad u'(0) = u_1 \in L^2(\Omega). \end{cases}$$

Let $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$ be a bounded open set, $f \in L^2(L^2)$,

$$\begin{cases} u'' - \Delta u = f & \text{in } (0,T] \times \Omega \\ u = 0 & \text{on } (0,T] \times \partial\Omega \\ u(0) = u_0 \in H_0^1(\Omega), \quad u'(0) = u_1 \in L^2(\Omega). \end{cases}$$

Given $X := H^2(H^{-1}) \cap L^2(H_0^1) \cap H^1(L^2)$, $Y := L^2(H_0^1)$, we consider the weak formulation: Find $u \in X$ such that for all $v \in Y$,

$$\int_0^T [\langle u'', v\rangle + (\nabla u, \nabla v)] = \int_0^T (f, v),$$

with $u(0) = u_0$ and $u'(0) = u_1$.

Typical example: ripples

Ripples on a pond,
from Rapideye, fineartamerica

▸ ripples

Typical example: ripples

Ripples on a pond,
from Rapideye, fineartamerica

▸ ripples

Question: How to efficiently capture local feature?

Typical example: ripples

Ripples on a pond,
from Rapideye, fineartamerica

▸ ripples

Question: How to efficiently capture local feature?
Our strategy: A posteriori analysis & space-time adaptive scheme

Explicit time stepping schemes

- very efficient, suitable for parallel computing.
- CFL condition $\tau \leq Ch_{\min}$ is needed.

See: Georgoulis, Lakkis, Makridakis, & Virtanen (SINUM '16), Chaumont-Frelet (SISC '23), Chaumont-Frelet & A. Ern (M2AN '25), Grote & Lakkis & Santos ('24, '25), Hugot, Imperiale, Vohralík ('26)...

Explicit time stepping schemes

- very efficient, suitable for parallel computing.
- CFL condition $\tau \leq C h_{\min}$ is needed.

See: Georgoulis, Lakkis, Makridakis, & Virtanen (SINUM '16), Chaumont-Frelet (SISC '23), Chaumont-Frelet & A. Ern (M2AN '25), Grote & Lakkis & Santos ('24, '25), Hugot, Imperiale, Vohralík ('26)...

Implicit time stepping schemes

- no CFL condition, ideal for space-time adaptive methods.
- expensive to solve the linear system.

See: Johnson (CMAME '93), Bernardi & Süli (M3AS '05), Bangerth & Rannacher (JCA. 01), Bangerth & Geiger & Rannacher (CMAM. 10), Georgoulis, Lakkis, & Makridakis (IMAJNA '13), Gorynina & Lozinski & Picasso (IMAJNA '19),...

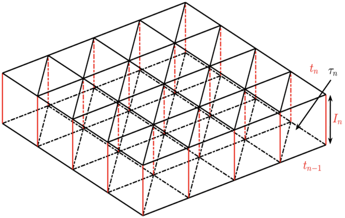
Space time FEM space

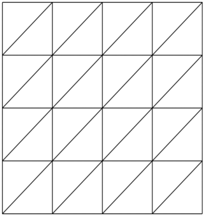
We seek U from

$$V_{p,\boldsymbol{q}} = \{W \in \mathbb{P}_{\boldsymbol{q}}(J; H_0^1(\Omega)) \ : \ W_{I_n} \in \mathbb{P}_{q_n}(I_n; V_h^n)\},$$

with $n \in \{1, ..., N\}$ and V_h^n is the H^1-conforming FEM space

$$V_h^n = \{v \in H_0^1(\Omega) \ : \ v|_K \in \mathbb{P}_p(K) \ \forall K \in \mathcal{T}_h^n\}.$$





Find $U \in V_{p,\boldsymbol{q}}$ such that for all $n \in \{1, ..., N\}$, $W \in \mathbb{P}_{q_n-1}(I_n; V_h^n)$,

$$\begin{cases} \int_{I_n} \{(U'', W) + (\nabla U, \nabla W)\} + ([U']_{n-1}, W(t_{n-1}^+)) = \int_{I_n} (f, W) \\ U(t_{n-1}^+) = \mathcal{P}_p^n U(t_{n-1}^-) \end{cases}$$

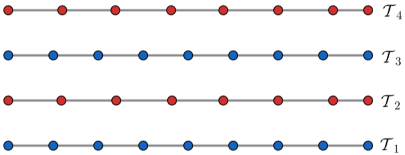
with $\mathcal{P}_p^n : V_h^{n-1} \to V_h^n$ the elliptic projector and the jump operator $[V]_{n-1} = V(t_{n-1}^+) - V(t_{n-1}^-)$. (cf. Walkington (SINUM. '14))

Find $U \in V_{p,\boldsymbol{q}}$ such that for all $n \in \{1, ..., N\}$, $W \in \mathbb{P}_{q_n-1}(I_n; V_h^n)$,

$$\begin{cases} \int_{I_n} \{(U'', W) + (\nabla U, \nabla W)\} + ([U']_{n-1}, W(t_{n-1}^+)) = \int_{I_n} (f, W) \\ U(t_{n-1}^+) = \mathcal{P}_p^n U(t_{n-1}^-) \end{cases}$$

with $\mathcal{P}_p^n : V_h^{n-1} \to V_h^n$ the elliptic projector and the jump operator $[V]_{n-1} = V(t_{n-1}^+) - V(t_{n-1}^-)$. (cf. Walkington (SINUM. '14))

- space-time FEM allows a posteriori analysis in time.
- $\mathcal{P}_p^n$ minimizes theoretical/practical mesh-change error.



Main idea: at each time step n,

1. find U^n
2. compute time estimator η^n_{time}, refine/coarsen I_n;
3. compute space estimator η^n_{space} and mesh-change estimator η^n_{mesh}, refine/coarsen $\mathcal{T}^n_h$;
4. repeat above steps until given tolerance is satisfied;
5. go to next time step.

We have the following error bound with traceable constant $C(p, \boldsymbol{q})$:

$$\|u - U\|_{L^\infty(L^2)} \le C(p, \boldsymbol{q})(\eta_{init} + \eta_f + \eta_{time} + \eta_{space} + \eta_{mesh}).$$

Denoting $\Delta_h^- = \Delta - \Delta_h$, the estimators are given by

$$\eta_{init} = \|u_0 - u_{0,h}\| + \|u_1 - u_{1,h}\| + \|h_0^2 \Delta_h^- u_{1,h}\| + \mathcal{J}(u_{1,h}, \mathcal{F}_I^1)$$

$$\eta_{time} = \sum_n \tau_n^3 (\|[\Delta_h U']_{n-1}\| + \|\Pi_{\boldsymbol{q}-1}^\perp \Delta_h U\|_{L^1(I_n;L^2)}) + \max_n \tau_n \|[U']_{n-1}\|$$

$$\eta_{space} = \sum_n \int_{I_n} \{\|h_n^2 \Delta_h^- U'\| + \mathcal{J}(U', \mathcal{F}_I^n)\} + \max_n \ldots + \ldots$$

$$\eta_{mesh} = \sum_n + \tau_n \tilde{\tau}_n \|[\Delta_h U]_{n-1}\| + \sum_n \{\|[U]_{n-1}\| + \tau_n^{-1} \|h_n^2 \Pi_p^\perp U'(t_{n-1}^-)\|\}$$
$$+ \max_n \ldots + \ldots$$

Error splitting

$$u - U = (u - \bar{U}) + (\bar{U} - U) = \rho_1 + \rho_2,$$

with $\bar{U} \in \mathcal{C}^1(\bar{J}; H_0^1(\Omega) \cap H^2(\Omega))$ the space-time reconstruction of U, be defined later.

- ρ_1 is controlled by PDE stability.
- ρ_2 is controlled by approximation properties.

Makridakis-Nochetto time reconstruction

The time reconstruction $\hat{V} \in \mathbb{P}_{q_n+1}(I_n; L^2(\Omega))$ of $V \in \mathbb{P}_{q_n}(I_n; L^2(\Omega))$ satisfies, for all $W|_{I_n} \in \mathbb{P}_{q_n-1}(I_n; L^2(\Omega))$,

$$\begin{cases} \int_{I_n}(\hat{V}'', W) = \int_{I_n}(V'', W) + ([V']_{n-1}, W(t_{n-1}^+)) \\ \hat{V}(t_{n-1}^+) = V(t_{n-1}^+) \quad \hat{V}'(t_{n-1}^+) = V'(t_{n-1}^-) \end{cases} .$$

See: Makridakis & Nochetto (Numer.Math. '06), Schötzau & Wihler (Numer.Math. '10), Holm & Wihler (Numer.Math. '17) and Dong & Mascotto & Wang (Accepted in Numer.Math. '26)

- we can reformulate the scheme as

$$\int_{I_n} \{(\hat{U}'', W) + (\nabla U, \nabla W)\} = \int_{I_n} (f, W).$$

- we can reformulate the scheme as

$$\int_{I_n} \{(\hat{U}'', W) + (\nabla U, \nabla W)\} = \int_{I_n} (f, W).$$

- We have computable bounds with traceable constant $C(q_n)$

$$\tau_n^{-3} \|V - \hat{V}\|^2_{L^2(I_n;L^2)} + \tau_n^{-1} \|V' - \hat{V}'\|^2_{L^2(I_n;L^2)}$$
$$+ \tau_n \|V'' - \hat{V}''\|^2_{L^2(I_n;L^2)} + \|V' - \hat{V}'\|^2_{L^\infty(I_n;L^2)} \leq C(q_n) \|[V']_{n-1}\|.$$

- we can reformulate the scheme as

$$\int_{I_n} \{(\hat{U}'', W) + (\nabla U, \nabla W)\} = \int_{I_n} (f, W).$$

- We have computable bounds with traceable constant $C(q_n)$

$$\tau_n^{-3} \|V - \hat{V}\|^2_{L^2(I_n;L^2)} + \tau_n^{-1} \|V' - \hat{V}'\|^2_{L^2(I_n;L^2)} \\ + \tau_n \|V'' - \hat{V}''\|^2_{L^2(I_n;L^2)} + \|V' - \hat{V}'\|^2_{L^\infty(I_n;L^2)} \leq C(q_n) \|[V']_{n-1}\|.$$

- $\hat{V}' \in \mathcal{C}^0(\bar{J}; L^2(\Omega))$, but $\hat{V} \notin \mathcal{C}^0(\bar{J}; L^2(\Omega))$ with mesh coarsening.

we consider the cubic Hermite polynomials $\{\phi_i^n\}_{i=1}^4$ with $\phi_1^n(t_{n-1}) = \phi_2^n(t_n) = (\phi_3^n)'(t_{n-1}) = (\phi_4^n)'(t_n) = 1$. Picking any basis $\{\hat{\phi}_i^n\}_{i=1}^{q_n-2}$ spanning $\mathbb{P}_{q_n-3}(I_n;\mathbb{R})$, we define the $\mathcal{C}^1$ bubble functions

$$\phi_i^n = \Big((t-t_{n-1})(t_n-t)\Big)^2 \hat{\phi}_{i-4}^n \qquad \forall i = 5, \dots q_n+2.$$

The set $\{\phi_i^n\}_{i=1}^{q_n+2}$ forms a basis of $\mathbb{P}_{q_n+1}(I_n;\mathbb{R})$.

Hermite-type reconstruction

For $V|_{I_n} = V(t_{n-1}^+)\phi_1^n + V(t_n^-)\phi_2^n + \sum_{i=3}^{q_n+2} V_i\phi_i^n$, we define the Hermite-type time reconstruction by

$$\mathcal{I}_{\mathfrak{c}}(V)|_{I_n} = \frac{V(t_{n-1}^-)+V(t_{n-1}^+)}{2}\phi_1^n + \frac{V(t_n^-)+V(t_n^+)}{2}\phi_2^n + \sum_{i=3}^{q_n+2} V_i\phi_i^n$$

- we can reformulate the scheme as

$$\int_{I_n} \{(\mathcal{I}_{\mathfrak{c}}(\hat{U})'', W) + (\nabla U, \nabla W)\} = \int_{I_n} (f + \mathcal{I}_{\mathfrak{c}}(\hat{U})'' - \hat{U}'', W).$$

- $\mathcal{I}_{\mathfrak{c}}(\hat{U}) \in \mathcal{C}^1(\bar{J}; H_0^1(\Omega))$.
- We have computable bounds with traceable constant C

$$\|V' - \mathcal{I}_{\mathfrak{c}}(V)'\|_{L^1(I_n;L^2)} + \tau_n^{-1}\|V - \mathcal{I}_{\mathfrak{c}}(V)\|_{L^1(I_n;L^2)} + \|V - \mathcal{I}_{\mathfrak{c}}(V)\|_{L^\infty(I_n;L^2)} \leq C(\|[V]_{n-1}\| + \|[V]_n\|).$$

- See Dong & Georgoulis & Mascotto & Wang (Accepted in Numer.Math. '26).

Elliptic reconstruction

we define its elliptic reconstruction $\tilde{U}$ of U, with $\tilde{U}|_{I_n} \in \mathbb{P}_{q_n}(I_n; H_0^1(\Omega) \cap H^2(\Omega))$, by

$$(\nabla\tilde{U}, \nabla w) = (g, w) \quad \forall w \in H_0^1(\Omega),$$

where $g = -\Delta_h U - (\Pi_{p,\boldsymbol{q}-1} - \Pi_{\boldsymbol{q}-1})f + \Pi_p^{\perp}\hat{U}''$.

See: Makridakis & Nochetto (SINUM '03), Lakkis & Makridakis (Math.Comp. '06), Georgoulis, Lakkis, & Makridakis (IMAJNA '13), Georgoulis, Lakkis, Makridakis, & Virtanen (SINUM '16), Georgoulis, Lakkis & Wihler (Numer.Math. '21), Georgoulis, & Makridakis (IMAJNA '23)...

- Recall $\bar{U} = \mathcal{I}_{\mathfrak{c}}(\hat{\tilde{U}}) \in \mathcal{C}^1(\bar{J}; H_0^1(\Omega) \cap H^2(\Omega))$, we can reformulate the scheme as

$$\int_{I_n} \{\bar{U}'', W + (\nabla \bar{U}, \nabla W)\} = \int_{I_n} (\text{nice terms}, W).$$

- We have computable bound for all $t \in I_n$, $k \in \mathbb{N}$,

$$\|\partial_t^k(\tilde{U}(t) - U(t))\| \leq \mathcal{E}(\partial_t^k U(t), \partial_t^k g(t); L^2(\Omega)) \leq \text{terms in estimators}.$$

Baker's argument

Inspired by Baker (SINUM '78), we set
$v \in \mathcal{C}^0(\bar{J}; H_0^1(\Omega)) \cap \mathcal{C}^1(\bar{J}; L^2(\Omega))$ by

$$v(t) = \int_t^\xi (u - \bar{U})(s)\mathrm{d}s,$$

with $\xi \in \bar{J}$ the point where $\|u(t) - \bar{U}(t)\|$ attains its maximum.

Baker's argument

Inspired by Baker (SINUM '78), we set $v \in \mathcal{C}^0(\bar{J}; H_0^1(\Omega)) \cap \mathcal{C}^1(\bar{J}; L^2(\Omega))$ by

$$v(t) = \int_t^\xi (u - \bar{U})(s)\mathrm{d}s,$$

with $\xi \in \bar{J}$ the point where $\|u(t) - \bar{U}(t)\|$ attains its maximum.

- Error equation can be reformulated as

$$\rho_1'' - \Delta\rho_1 = \Pi_{\boldsymbol{q}-1} f + (\bar{U} - \hat{U})'' + \Delta(\tilde{U} - \bar{U}) - \Pi_{\boldsymbol{q}-1}^{\perp}\Delta_h U.$$

Baker's argument

Inspired by Baker (SINUM '78), we set $v \in \mathcal{C}^0(\bar{J}; H_0^1(\Omega)) \cap \mathcal{C}^1(\bar{J}; L^2(\Omega))$ by

$$v(t) = \int_t^\xi (u - \bar{U})(s)\mathrm{d}s,$$

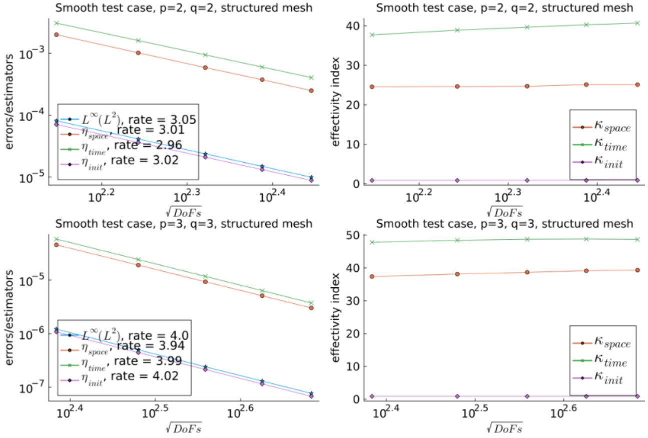
with $\xi \in \bar{J}$ the point where $\|u(t) - \bar{U}(t)\|$ attains its maximum.

- Error equation can be reformulated as

$$\rho_1'' - \Delta\rho_1 = \Pi_{\boldsymbol{q}-1} f + (\bar{U} - \hat{U})'' + \Delta(\tilde{U} - \bar{U}) - \Pi_{\boldsymbol{q}-1}^{\perp} \Delta_h U.$$

- Testing with Baker's test function v, estimating terms in RHS complete the proof.

For a smooth test, we have optimal convergence order with uniform refinement:



Numerical test: water drop

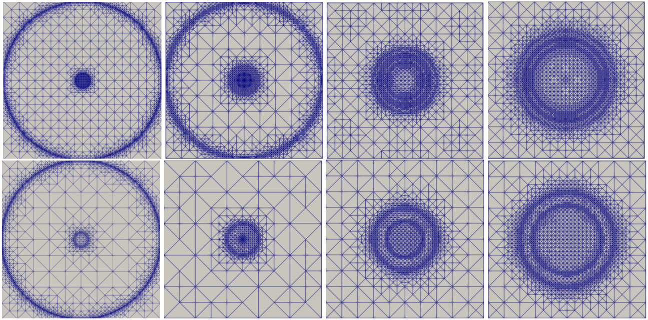
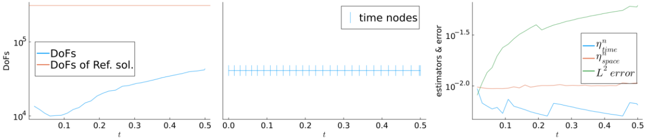


Figure 6: Visualization of dynamic mesh modification of the test case with exact solution with data as in (6.2) computed with Algorithm 1; $q = 2$; p equals 1 and 2 for top figures and bottom figures, respectively; T equals 0, 0.1, 0.3 and 0.5 from left to right.

Numerical test: water drop



▸ ripples

References

1. Z. Dong, E. H. Georgoulis, L. Mascotto, and Z. Wang,
   A posteriori error analysis and adaptivity of a space–time finite element method for the wave equation in second-order formulation,
   accepted for publication in Numerische Mathematik, 2026; arXiv:2509.08537.

2. Z. Dong, L. Mascotto, and Z. Wang,
   A priori and a posteriori error estimates of a C^0-in-time method for the wave equation in second-order formulation,
   accepted for publication in Numerische Mathematik, 2026; arXiv:2411.03264.

Thank you for your attention!