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Bound-preserving & mass-conservative high-order methods for fourth-order elliptic problems and nonlinear parabolic equations

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A phase-field model for phase separation [Cahn & Hilliard, J. Chem. Phys., 1958]:

$$\partial_t u = \nabla \cdot (M(u) \nabla w), \quad w = -\Delta u + f(u), \quad f = F'.$$

u the phase variable, $M(u) = 1 - u^2 \geq 0$ the mobility,
 $f(u) = \log(1 + u) - \log(1 - u) - 5u$ the potential derivative.

- Bound-preservation:

$$-1 < u(x, t) < 1.$$

- Mass-conservation:

$$\int_D u(t) = \int_D u_0.$$

- Energy dissipation:

$$\frac{d}{dt} E(u) = - \int_D M(u) |\nabla w|^2 \leq 0, \quad E(u) = \frac{1}{2} \|\nabla u\|^2 + \int_D F(u).$$

A fourth-order model for a thin liquid film [Bertozzi & Pugh, CPAM, 1996; Zhornitskaya & Bertozzi, SINUM, 2000]:

$$\partial_t u = \nabla \cdot (u^\beta \nabla w), \quad w = -\Delta u, \quad \beta > 0.$$

u the film thickness, $M(u) = u^\beta$, $\beta > 0$ a degenerate mobility.

- Bound-preservation:

$$u(x, t) \geq 0.$$

- Mass-conservation:

$$\int_D u(t) = \int_D u_0.$$

- Energy dissipation:

$$\frac{d}{dt} \left(\frac{1}{2} \|\nabla u\|^2 \right) = - \int_D u^\beta |\nabla w|^2 \leq 0.$$

A nonlinear diffusion equation with density-dependent diffusivity [Vázquez, *The Porous Medium Equation*, Oxford University Press, 2007]:

$$\partial_t u = \Delta(u^m) = m \nabla \cdot (u^{m-1} \nabla u), \quad m > 1.$$

u a nonnegative density, the diffusion degenerates in the vacuum region.

- Bound-preservation:

$$u_0 \geq 0 \quad \implies \quad u(x, t) \geq 0.$$

- Mass-conservation:

$$\int_D u(t) = \int_D u_0.$$

- Energy dissipation:

$$\frac{d}{dt} \left(\frac{1}{2} \|u\|^2 \right) = -m \int_D u^{m-1} |\nabla u|^2 \leq 0.$$

Bound-preservation $a \leq U_i^n \leq b,$

Mass-conservation $\int_D u_h^n = \int_D u_h^0,$

Energy dissipation $E(u_h^{n+1}) \leq E(u_h^n).$

Can a high-order method preserve some of them?

1. Stationary linear fourth-order elliptic problems
2. Nonlinear parabolic problems

- Discrete maximum principles

Low order, mesh restrictions.

Barrenechea, John & Knobloch, SIAM Rev., 2024.

- Variational inequalities

High-order bound-preservation. Mass-conservation is not automatic.

Barrenechea et al., IMA J. Numer. Anal., 2024.

Kirby & Shapero, Numer. Math., 2024.

- Bound-preservation + mass conservation

Enriched Galerkin methods, optimization, redistribution.

Barrenechea, Lederer & Rupp, 2025. Liu et al., SIAM J. Sci. Comput., 2024.

Guermond & Wang, J. Comput. Phys., 2025.

- Energy stability

Elliott & Stuart, SINUM, 1993. Shen & Yang, DCDS, 2010.

Shen, Xu & Yang, SIAM Rev., 2019. Cheng, Liu & Shen, CMAME, 2020.

- Bound-preservation and mass-conservation

Cheng & Shen, CMAME, 2022. Cheng & Shen, SINUM, 2022.

Tong & Cai, SINUM, 2024

- Bound-preservation and energy stability

Huang & Shen, SISC, 2021. Cheng, Wang & Zhao, preprint, 2025.

Fourth-order elliptic problems

High-order finite elements with
bound-preservation and mass-conservation

Time-dependent problems

High-order BDF-FEM schemes with
bound-preservation and mass-conservation
SAV stabilization for improved energy behavior

Preprint:

<https://arxiv.org/abs/2605.23486>

<https://hal.science/hal-05630475v1>

Fourth-order elliptic problems

Let $D \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$. Find (u, w) such that

$$\begin{cases} u = \nabla \cdot (M \nabla w) + f_1, \\ w = -\nabla \cdot (\kappa \nabla u) + f_2, \end{cases} \quad \text{in } D,$$

with homogeneous Neumann boundary conditions.

Assumptions

$$\begin{aligned} 0 < M_1 \leq M \leq M_2, \quad 0 < \kappa_1 \leq \kappa \leq \kappa_2, \\ f_1, f_2 \in L^2(D), \quad u(x) \in [a, b]. \end{aligned}$$

Define

$$\mathcal{K} := \{v \in H^1(D) : a \leq v \leq b \text{ a.e.}\}.$$

Find $(u, w) \in \mathcal{K} \times H^1(D)$ such that

$$\begin{cases} (u, v) + (M \nabla w, \nabla v) = (f_1, v) & \forall v \in H^1(D), \\ (w, \xi - u) \leq (\kappa \nabla u, \nabla(\xi - u)) + (f_2, \xi - u) & \forall \xi \in \mathcal{K}. \end{cases}$$

The constraint $u \in [a, b]$ is built into $\mathcal{K}$.

The mass-conservation is built by taking $v = 1$.

See [Han ,2024, Ch. 3]; [Blank, Butz & Garcke, 2011].

Let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular mesh of D , and

$$V_h^p := \text{span}\{\phi_i\}_{i \in \mathcal{V}}, \quad \mathcal{V} := \{1:I\}.$$

$$V_h^{p,+} := \left\{ v_h = \sum_{i \in \mathcal{V}} V_i \phi_i \in V_h^p \mid V_i \in [a, b], \forall i \in \mathcal{V} \right\}.$$

Find $(u_h, w_h) \in V_h^{p,+} \times V_h^p$ such that

$$\begin{cases} (u_h, v_h) + (M \nabla w_h, \nabla v_h) = (f_1, v_h), & \forall v_h \in V_h^p, \\ (w_h, \xi_h - u_h) \leq (\kappa \nabla u_h, \nabla (\xi_h - u_h)) + (f_2, \xi_h - u_h), & \forall \xi_h \in V_h^{p,+}. \end{cases}$$

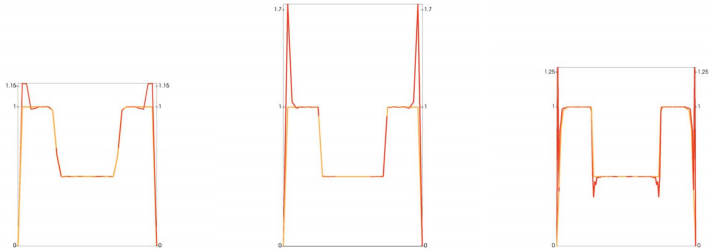
No acute-angle condition [Barrenechea, John & Knobloch, 2024]

Solver: PDAS (semismooth Newton) [Hintermüller, Ito & Kunisch, 2002]

Related VI methods: [Blank, Butz & Garcke, 2011]; [Barrenechea, Georgoulis & Pryer, 2023]; [Kirby & Shapero, 2024]; [Dong, Ern & Wang, 2025].

Oscillation can be significantly alleviated

[Barrenechea, Georgoulis & Pryer, 2023]



(A) $\epsilon = 10^{-4}$, $k = 1$.

(B) $\epsilon = 10^{-7}$, $k = 1$.

(C) $\epsilon = 10^{-7}$, $k = 2$.

FIG. 7. A cross section taken about the $x = y$ plane of the finite element solution, coloured red presenting oscillations at the boundary layer, and the approximation given by (5.3), coloured green with no oscillations at the boundary. Panels (A) and (B) are computed using $k = 1$, while (C) depicts the approximations obtained using $k = 2$.

Define the admissible set

$$Y := \left\{ v_h \in V_h^{p,+} : \int_D v_h = \int_D f_1 \right\}.$$

The component u_h is the unique minimizer of

$$\min_{v_h \in Y} \left\{ \frac{1}{2} \|\sqrt{\kappa} \nabla v_h\|^2 + (f_2, v_h) + \frac{1}{2} \|v_h - f_1\|_{H_h^{-1}}^2 \right\}.$$

Consequences

- existence and uniqueness of u_h ,
- bound-preservation,
- mass-conservation.

Here $\|g\|_{H_h^{-1}}^2 = (g, (-\Delta_h^M)^{-1}g)$.

Assume that D has a sufficiently smooth boundary and

$$u \in C^0(\overline{D}; [a, b]) \cap W^{p, \infty}(D) \cap H^{p+1}(D), \quad w \in H^{p+1}(D).$$

For $h < h_0(u)$,

$$\|\nabla(u - u_h)\| \leq Ch^p.$$

Two ingredients:

1. Construct an optimal admissible candidate:

$$\Pi_h u \in Y, \quad \|\nabla(\Pi_h u - u)\| \leq Ch^p.$$

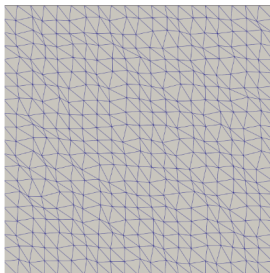
2. Apply standard FEM consistency and stability arguments to the H^1 - H^{-1} representation of the scheme.

Consider

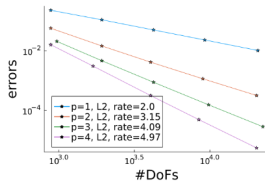
$$\begin{cases} u = \nabla \cdot ((1+x)\nabla w) + f_1, \\ w = -\Delta u, \end{cases} \quad D = (0,1)^2,$$

with

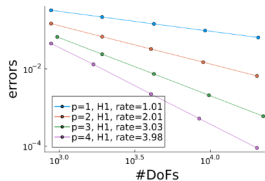
$$u(x, y) = \cos(4\pi x) \cos(4\pi y).$$



(a) Mesh with obtuse triangles.



(b) Convergence test, L^2 -norm.



(c) Convergence test, H^1 -norm.

Application on parabolic problems

Consider [Bertozzi & Pugh, 1996; Zhornitskaya & Bertozzi, 2000]

$$\begin{cases} \partial_t u = \nabla \cdot (M(u) \nabla w), & \text{in } D \times (0, T], \\ w = -\Delta u, & \text{in } D \times (0, T], \\ \partial_{\mathbf{n}} u = \partial_{\mathbf{n}} w = 0, & \text{on } \partial D \times (0, T], \\ u(\cdot, 0) = u_0. \end{cases}$$

For $k \in \{1, \dots, 5\}$, find

$$(u_h^{n+1}, w_h^{n+1}) \in V_h^{p,+} \times V_h^p$$

such that

$$\begin{cases} \left(\frac{\alpha_k u_h^{n+1} - A_k(u_h^n)}{\tau}, v_h \right) + (M(\bar{u}_h^{n+1}) \nabla w_h^{n+1}, \nabla v_h) = 0 & \forall v_h \in V_h^p \\ (w_h^{n+1}, \xi_h - u_h^{n+1}) \leq (\nabla u_h^{n+1}, \nabla (\xi_h - u_h^{n+1})) & \forall \xi_h \in V_h^{p,+} \end{cases}$$

Here (α_k, A_k) denotes BDF k , and

$$\bar{u}_h^{n+1} := \sum_{i \in \mathcal{V}} \phi_i \min\{b, \max\{a, B_k(U_i^n)\}\},$$

where B_k is the standard extrapolation operator.

Explicit mobility, implicit variational inequality.

For every time step, a solution (u_h^{n+1}, w_h^{n+1}) exists, and u_h^{n+1} is unique. Moreover,

$$U_i^{n+1} \in [a, b] \quad \forall i \in \mathcal{V},$$

and

$$\int_D u_h^{n+1} = \int_D u_h^n.$$

In fact,

$$u_h^{n+1} = \arg \min_{v \in Y^{n+1}} \left\{ \frac{\alpha_k}{2} \|\nabla v\|^2 + \frac{1}{2\tau} \|\alpha_k v - A_k(u_h^n)\|_{H_h^{-1, n+1}}^2 \right\},$$

where

$$Y^{n+1} := \left\{ v \in V_h^{p,+} : \int_D v = \int_D u_h^n \right\}.$$

For BDF1,

$$\frac{1}{2} \|\nabla u_h^{n+1}\|^2 \leq \frac{1}{2} \|\nabla u_h^n\|^2.$$

Energy stability is proved for BDF1 only.

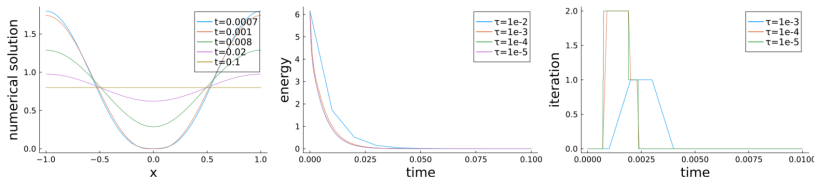
Consider [Cheng & Shen, 2022; Zhornitskaya & Bertozzi, 2000]

$$\begin{cases} \partial_t u = \nabla \cdot (\sqrt{u} \nabla w), \\ w = -\Delta u, \end{cases} \quad D = (-1, 1),$$

with $u_0(x) = 0.8 - \cos(\pi x) + 0.25 \cos(2\pi x)$.

The solution develops a temporal singularity near $t_\star \approx 7.4 \times 10^{-4}$.

50 nodes, $p = k = 1$.



(a) Numerical solution, $\tau = 10^{-4}$. (b) Energy evolution for different τ . (c) Iteration number for different τ .

Stable through the singularity without mobility regularization.

Consider [Cahn & Hilliard, 1958]

$$\begin{cases} \partial_t u = \nabla \cdot (M(u) \nabla w), & \text{in } D \times (0, T], \\ w = -\Delta u + f(u), & \text{in } D \times (0, T], \\ \partial_{\mathbf{n}} u = \partial_{\mathbf{n}} w = 0, & \text{on } \partial D \times (0, T], \\ u(\cdot, 0) = u_0, \end{cases}$$

Introduce [Shen, Xu & Yang, 2019]

$$r(t) := \sqrt{E_1(u(t))}.$$

For $k \in \{1, \dots, 5\}$, find $(u_h^{n+1}, r^{n+1}, w_h^{n+1}) \in V_h^{p,+} \times \mathbb{R} \times V_h^p$ such that

$$\left\{ \begin{array}{l} \left(\frac{\alpha_k u_h^{n+1} - A_k(u_h^n)}{\tau}, v_h \right) + (M(\bar{u}_h^{n+1}) \nabla w_h^{n+1}, \nabla v_h) = 0 \quad \forall v_h \in V_h^p \\ (w_h^{n+1}, \xi_h - u_h^{n+1}) \leq (\nabla u_h^{n+1}, \nabla(\xi_h - u_h^{n+1})) \\ \quad + \left(\frac{r^{n+1} f(\bar{u}_h^{n+1})}{\sqrt{E_1(\bar{u}_h^{n+1})}}, \xi_h - u_h^{n+1} \right) \quad \forall \xi_h \in V_h^{p,+} \\ \alpha_k r^{n+1} - A_k(r^n) = \left(\frac{f(\bar{u}_h^{n+1})}{2\sqrt{E_1(\bar{u}_h^{n+1})}}, \alpha_k u_h^{n+1} - A_k(u_h^n) \right) \end{array} \right.$$

Here $\bar{u}_h^{n+1}$ is the clipped extrapolation defined for the lubrication scheme.

Explicit nonlinear terms and one scalar auxiliary variable.

For every time step, the scheme admits a solution, and u_h^{n+1} is unique. The pair (u_h^{n+1}, r^{n+1}) is the unique minimizer of

$$\min_{(v,s) \in X^{n+1}} \left\{ \frac{\alpha_k}{2} \|\nabla v\|^2 + \alpha_k s^2 + \frac{1}{2\tau} \|\alpha_k v - A_k(u_h^n)\|_{H_h^{-1,n+1}}^2 \right\}.$$

Therefore,

$$U_i^{n+1} \in [a, b] \quad \forall i \in \mathcal{V},$$

and

$$\int_D u_h^{n+1} = \int_D u_h^n.$$

For BDF1,

$$\frac{1}{2} \|\nabla u_h^{n+1}\|^2 + (r^{n+1})^2 \leq \frac{1}{2} \|\nabla u_h^n\|^2 + (r^n)^2.$$

Modified energy stability is proved for BDF1 only.

Consider [Cheng & Shen, 2022]

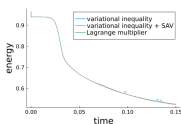
$$\begin{cases} \partial_t u = \nabla \cdot ((1 - u^2) \nabla w), \\ w = -0.01 \Delta u + \log(1 + u) - \log(1 - u) - 5u, \end{cases} \quad D = (0, 1)^2,$$

with

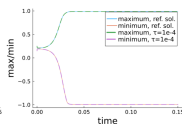
$$u_0(x, y) = 0.2 + 0.05 \operatorname{rand}(-1, 1), \quad T = 1.5 \times 10^{-1}.$$

The solution must remain in $(-1, 1)$, $M(u) = 1 - u^2 \rightarrow 0$ as $u \rightarrow \pm 1$.

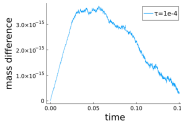
$$C_0 = 1, \quad p = 1, \quad k = 2, \quad 50 \times 50 \text{ nodes.}$$



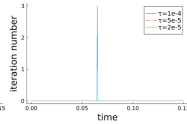
(a) Comparison of energy evolution, $\tau = 10^{-4}$.



(b) Maximum and minimum of u_h^n .



(c) Mass-conservation of u_h^n .



(d) Comparison of iteration number.

Consider

$$\begin{cases} \partial_t u = \nabla \cdot (K(u) \nabla u), & \text{in } D \times (0, T], \\ \partial_{\mathbf{n}} u = 0, & \text{on } \partial D \times (0, T], \\ u(\cdot, 0) = u_0. \end{cases}$$

Introduce the fourth-order regularization

$$\begin{cases} \partial_t u = \nabla \cdot (K(u) \nabla u) + \sqrt{\varepsilon} \Delta w, \\ w = -\sqrt{\varepsilon} \Delta u. \end{cases}$$

The perturbation error is formally $\mathcal{O}(\varepsilon)$.

Set

$$\varepsilon = h^{p+1}.$$

For $k \in \{1, \dots, 5\}$, find $(u_h^{n+1}, w_h^{n+1}) \in V_h^{p,+} \times V_h^p$ such that

$$\begin{cases} \left(\frac{\alpha_k u_h^{n+1} - A_k(u_h^n)}{\tau}, v_h \right) + (K(\bar{u}_h^{n+1}) \nabla u_h^{n+1}, \nabla v_h) \\ \quad + \sqrt{h^{p+1}} (\nabla w_h^{n+1}, \nabla v_h) = 0, & \forall v_h \in V_h^p, \\ (w_h^{n+1}, \xi_h - u_h^{n+1}) \leq \sqrt{h^{p+1}} (\nabla u_h^{n+1}, \nabla (\xi_h - u_h^{n+1})), & \forall \xi_h \in V_h^{p,+}. \end{cases}$$

Assume that the discrete problem admits a solution.

Then

$$U_i^{n+1} \in [a, b] \quad \forall i \in \mathcal{V},$$

and

$$\int_D u_h^{n+1} = \int_D u_h^n.$$

Well-posedness is not covered by the current analysis.

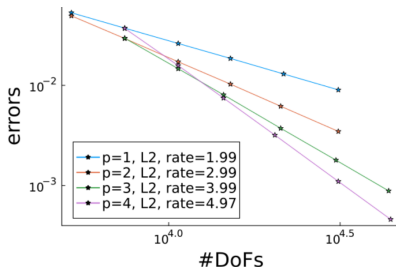
Numerical experiments indicate stable solvability.

Consider

$$\partial_t u = \nabla \cdot ((1 + u) \nabla u) + f_1, \quad D = (0, 2\pi)^2,$$

with $u(x, y, t) = (\cos(x) \cos(y) - \frac{3}{8\pi} x^4 + x^3) \cos(t)$, $T = 1$.

$$\tau = \frac{h}{2}, \quad k = p + 1, \quad \partial_{\mathbf{n}} w_h = 0 \text{ although } \partial_{\mathbf{n}} w \neq 0.$$



Optimal L^2 -rates for $p = 1, \dots, 4$.

Consider [Cheng & Shen, 2022; Vázquez, 2007]

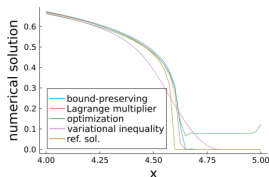
$$\partial_t u = m \nabla \cdot (u^{m-1} \nabla u), \quad D = (-5, 5), \quad T = 1.2,$$

The Barenblatt solution is

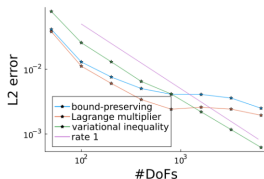
$$u(x, t) = \frac{1}{t_0^\alpha} \left[\max \left(0, 1 - \alpha \frac{m-1}{2m} \frac{x^2}{t_0^{2\alpha}} \right) \right]^{\frac{1}{m-1}}, \quad t_0 = t + 1, \quad \alpha = \frac{1}{m+1}.$$

The solution has compact support, moving interface, and $u \in H^s(D)$ with $s = 2, 1/2 + 1/(m-1)$ for $m = 2$ and $m \geq 3$, respectively.

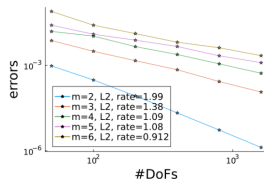
$$p = 1, \quad k = 2, \quad \tau = \frac{h}{10}.$$



(a) Numerical solution produced by different methods, $m = 6$.



(b) Convergence test for different methods, $m = 6$.



(c) Convergence test for the proposed method.

Linear fourth-order elliptic problems

high-order FEM + variational inequality

bound-preservation, mass-conservation, $\|\nabla(u - u_h)\| \leq Ch^p$.

Nonlinear fourth-order parabolic problems

BDF + variational inequality + SAV stabilization

bound-preservation, mass-conservation, energy stability for $k = 1$.

Nonlinear second-order parabolic problems

fourth-order regularization + BDF + variational inequality

bound-preservation and mass-conservation.

1. Time-dependent problems

well-posedness and error estimates

2. A posteriori error analysis and adaptivity

adaptive spatial and temporal discretizations

3. Nonlinear elliptic problems

analysis and high-order discretizations

Thank you for your attention!

Homepage:

<https://wangzuodong1997.github.io/>

Submitted manuscript:

<https://arxiv.org/abs/2605.23486>

<https://hal.science/hal-05630475v1>