Monotonicity, sonic points and convergence of a stabilized FEM scheme for scalar hyperbolic conservation laws

Finite element Rodeo 2024

Jean-Luc Guermond (Texas A&M university)
Bojan Popov (Texas A&M university)
★ Zuodong Wang (Ecole des Ponts ParisTech & Centre Inria de Paris)

PDE study

PDE (neglect the boundary condition)

École des Ponts
ParisTech

$$\partial_t u + \nabla\cdot\boldsymbol{f}(u) = 0, \qquad (\boldsymbol{x}, t) \in \Omega \times (0, T], \tag{1}$$

$$u(\cdot, 0) = u_0 \in BV. \tag{2}$$

- Even if $u_0 \in C^\infty$, $u(\cdot, t) \in L^\infty$ only.
- Maximum principle: $u_0(\boldsymbol{x}) \in [a, b] \Longrightarrow u(\boldsymbol{x}, t) \in [a, b]$.
- Entropy inequality: $\partial_t \eta(u) + \nabla\cdot\boldsymbol{q}(u) \leq 0$.

Numerical scheme

Basic idea

Vanishing viscosity [1]

$$\partial_t u^\epsilon + \nabla \cdot \boldsymbol{f}(u^\epsilon) = \epsilon \Delta u^\epsilon, \epsilon > 0 \Rightarrow u^\epsilon \to u, \text{ as } \epsilon \to 0.$$

[1] S. N. Kružkov, [1970], First order quasilinear equations in several independent variables

Stabilized FEM scheme[2]

$$m_i \frac{U_i^{n+1} - U_i^n}{\Delta t} + \sum_{j \in \mathcal{I}(i)} \boldsymbol{f}(U_j^n) c_{ij} = \sum_{j \in \mathcal{I}(i)} d_{ij}^n (U_j^n - U_i^n),$$

$$m_i = \int_\Omega \phi_i, \quad c_{ij} = \int_\Omega \nabla \phi_j \phi_i, \quad d_{ij}^n = \lambda_{ij}^n \max(\|c_{ij}\|, \|c_{ji}\|),$$

with CFL condition $\Delta t \leq \rho C h$, where $\rho \in (0, 1]$.

Remark

- $\lambda_{ij}^n \approx \epsilon = \epsilon(\boldsymbol{x}, t, u)$ at the continuous level
- λ_{ij}^n could be the local Lipschitz constant $|\boldsymbol{f}|_{W^{1,\infty}(stencil)}$
- λ_{ij}^n could be zero at sonic points ($\boldsymbol{f}'(u) = 0$).

[2] J.-L. Guermond and M. Nazarov [2014], A maximum-principle preserving C^0 finite element method for scalar conservation equations

Convergence analysis

Convergence in multi-dimension [3]

Basic idea for explicit schemes

1. $U_i^{n+1} = Conv_{j \in \mathcal{I}(i)}(U_j^n)$
2. establish the L^2-stability $\|\nabla u_h\|_{L^2(L^2)} \leq Ch^{-\frac{1}{2}}$.
3. establish entropy residual estimate: $\partial_t \eta(u_h) + \nabla \cdot \boldsymbol{q}(u_h) \leq Ch \int_0^T \|\nabla u_h(t)\|_{L^1}$
4. show the convergence by either the compensated compactness theorem or doubling variable technique (Kuznetsov's lemma)

Remark: This scheme is not monotone.

[3] J.-L. Guermond AND B. Popov, [2016], Error estimates of a first-order Lagrange finite element technique for nonlinear scalar conservation equations

Previously proposed idea for L^2-stability

Reminder: CFL condition: $\Delta t \leq \rho C h$.

L^2-stability

Assume that

1. ρ is sufficiently small (but still be $O(1)$),
2. $\lambda_{ij}^n > c_0$ (unnecessary diffusion at sonic points),

then we have

$$\|u_h^N\|_{L^2}^2 + Ch\|\nabla u_h\|_{L^2(L^2)}^2 \leq \|u_h^0\|_{L^2}^2.$$

Proof. Multiplying the scheme by U_i^{n+1}, involving the classical energy argument for parabolic PDEs. the lower bound on λ_{ij}^n gives the norm equivalence between viscosity and H^1 seminorm.

A tighter discrete entropy inequality I

Continuous level (formally):

$$\eta' \times \left(u_t + \nabla \cdot \boldsymbol{f} = \epsilon \Delta u\right) \Rightarrow \eta_t + \nabla \cdot \boldsymbol{q} = \epsilon \Delta\big(\eta(u)\big) - \epsilon \nabla \eta'(u) \cdot \nabla u,$$

where $\nabla \eta'(u_h^n) \cdot \nabla u_h^n \geq 0$ for $\eta(v) = \frac{1}{2} v^2$.

Discrete level (previously proposed version):

$$m_i \frac{\eta(U_i^{n+1}) - \eta(U_i^n)}{\Delta t} + \sum_j c_{ij} \boldsymbol{q}(U_j^n) \leq \sum_j d_{ij}^n \left(\eta(U_j^n) - \eta(U_i^n)\right)$$

$$\approx \epsilon \Delta\big(\eta(u_h^n)\big) - 0,$$

A tighter discrete entropy inequality II

École des Ponts ParisTech

Lemma (novel discrete entropy inequality)

For any $\rho \in (0,1]$, if the entropy η is strongly convex with parameter $\mu > 0$, we have

$$m_i \frac{\eta(U_i^{n+1}) - \eta(U_i^n)}{\Delta t} + \sum_j c_{ij} \boldsymbol{q}(U_j^n) + \frac{\mu(1-\rho)}{2} \sum_j d_{ij}^n (U_i^n - \overline{U}_{ij}^n)^2 \leq \sum_j d_{ij}^n (\eta(U_j^n) - \eta(U_i^n)).$$

Proof. In the previously proposed proof, invoke the inequality

$$\eta(\theta v + (1-\theta)w) \leq \theta\eta(v) + (1-\theta)\eta(w) - \frac{\mu}{2}\theta(1-\theta)(v-w)^2$$

instead of

$$\eta(\theta v + (1-\theta)w) \leq \theta\eta(v) + (1-\theta)\eta(w).$$

Relaxation of λ_{ij}^n (not the best idea!)

École des Ponts ParisTech

1. Replace λ_{ij}^n by $\tilde{\lambda}_{ij}^n := \max(h^\theta, \lambda_{ij}^n), \theta \in (0, \frac{1}{2})$.
2. Get a weaker BV estimate: $\|\nabla u_h^n\|_{L^2(L^2)} \leq Ch^{-(\frac{1}{2}+\theta)}$
3. $\|u_h - u\|_{L^\infty(L^1)} \to 0$, since the entropy residual still converges to zero.

That's all

Thank you!