Asymptotic-preserving and invariant-domain preserving schemes for scalar hyperbolic conservation laws with stiff source term

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PDE study

$$
\partial_t u^\varepsilon + \nabla \cdot \boldsymbol{f}(u^\varepsilon) = \frac{1}{\varepsilon} R(u^\varepsilon), \qquad (\boldsymbol{x}, t) \in \Omega \times (0, T], \tag{1}
$$

$$
u^\varepsilon(\cdot, 0) = u_0. \tag{2}
$$

- Invariant-domain preserving:
  $u_0(\boldsymbol{x}) \in [0,1] \Longrightarrow u^\varepsilon(\boldsymbol{x}, t) \in [0,1]$
- Entropy inequality: $\partial_t \eta(u^\varepsilon) + \nabla \cdot \boldsymbol{q}(u^\varepsilon) \leq \frac{1}{\varepsilon} \eta'(u^\varepsilon) R(u^\varepsilon)$
- (Assume)There exists a unique limit $u^0 := \lim_{\varepsilon \to 0} u^\varepsilon$

Flux examples

- Linear transport: $\boldsymbol{f}(v) = v$
- Burgers: $\boldsymbol{f}(v) = \frac{1}{2}v^2$
- KPP: $\boldsymbol{f}(v) = (\sin(v), \cos(v))$

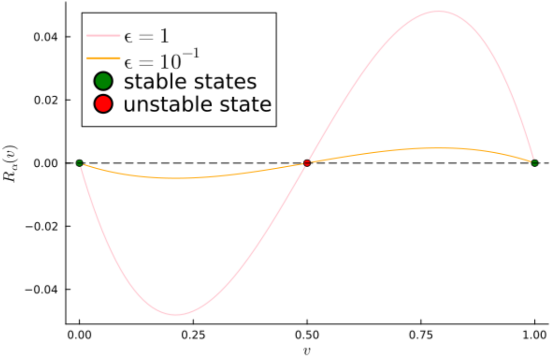


Figure: $R(v) = v(1-v)(v-\frac{1}{2})$

Disspative source (1 stable equilibrium point)

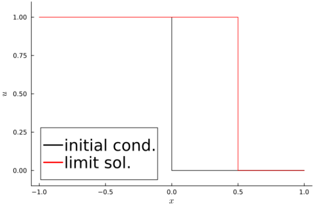
$$\partial_t u + \nabla \cdot \boldsymbol{f}(u) = \frac{1}{\varepsilon} R(u), \quad R(u) \leq 0$$

Yee-LeVeque model (2 stable equilibrium points)

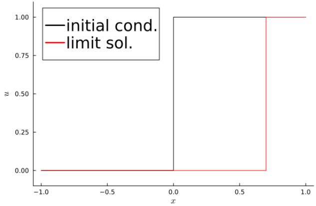
$$\partial_t u + \partial_x u = \frac{1}{\varepsilon} u(1-u)(u-\frac{1}{2})$$

Three stable equilibrium points

$$\partial_t u + \nabla \cdot \boldsymbol{f}(u) = \frac{1}{\varepsilon} u(1-u)(u-\frac{1}{2})(u-\frac{1}{4})(u-\frac{3}{4})$$



(a) shock-type initial condition



(b) rarefaction-type initial condition

Figure: $\boldsymbol{f}(v) = \frac{1}{2}v^2$, $T = 1$, $R(v) = v(1-v)(v-0.7)$

Shock: $\frac{\boldsymbol{f}(1)-\boldsymbol{f}(0)}{1-0} = 0.5$ Rarefaction: $\boldsymbol{f}'(\alpha) = 0.7$

- H. Fan, S. Jin, and Z.-H. Teng (2000)

General case?

We will know one day :)

02

Numerical study

- Invariant-domain-preserving: $u_h^\varepsilon(\boldsymbol{x}, t) \in [0, 1]$

- Asymptotic-preserving:
$\lim_{\varepsilon\to 0} \lim_{h\to 0} u_h^\varepsilon = u^0 = \lim_{h\to 0} \lim_{\varepsilon\to 0} u_h^\varepsilon$

- Entropy inequality:
$\langle \partial_t \eta(u_h^\varepsilon) + \nabla \cdot \boldsymbol{q}(u_h^\varepsilon) - \frac{1}{\varepsilon}\eta(u_h^\varepsilon)' R(u_h^\varepsilon), \psi \rangle \leq \Lambda(h) \to 0$

R. J. LeVeque and H. C. Yee (1990)

Classical ideas cannot correctly reproduce the correct shock location, unless $h = \mathcal{O}(\varepsilon)$.

W. Bao and S. Jin (2002)

- $\mathcal{O}(h \log h)$ convergence rate
- 1D, convex flux, shock-type initial condition

M. Svärd and S. Mishra (2011)

- Convergence is observed for both shock- and rarefaction-type initial conditions
- 1D, convex flux, no interaction between rarefaction and shock.

$$u_h^{n,\varepsilon} \xrightarrow{\text{transport}} w_h^{n+1,\varepsilon} \xrightarrow{\text{reaction}} u_h^{n+1,\varepsilon}$$

- Transport ($\frac{W_i^{n+1}-W_i^n}{\Delta t} + \nabla \cdot \boldsymbol{f}(w_h^n) = 0$):
  Stabilized FEM, upwind, Lax-Friedrichs, Godunov, ...

- Reaction ($u_h^{n+1} = v(\Delta t) : d_t v = \frac{1}{\varepsilon} R(v),\ v(0) = w_h^{n+1,\varepsilon}$):
  $d_t v = \frac{1}{\varepsilon} v(1-v)\tilde{R}(w_h^{n+1,\varepsilon}),\ \tilde{R}(w) = \frac{R(w)}{w(1-w)},\ v(0) = w_h^{n+1,\varepsilon}$

- $u_h^{n,\varepsilon}(\boldsymbol{x}) \in [0,1] \Rightarrow w_h^{n+1,\varepsilon}(\boldsymbol{x}) \in [0,1] \Rightarrow u_h^{n+1,\varepsilon}(\boldsymbol{x}) \in [0,1]$

Theorem (Entropy-inequality)

Assume that $\psi \in W_0^{1,\infty}$, *we have*

$$\langle \partial_t \eta(u_h^\varepsilon) + \nabla \cdot \boldsymbol{q}(u_h^\varepsilon) - \frac{1}{\varepsilon}\eta(u_h^\varepsilon)' R(u_h^\varepsilon), \psi \rangle \leq C\Big(\frac{h}{\varepsilon^2} \|u_h^\varepsilon\|_{L^1(L^1)} + \frac{h^2 + \varepsilon h}{\varepsilon^2} \|\nabla u_h^\varepsilon\|_{L^1(L^1)} \Big)$$

- Fix ε, manipulate h ✗
- Fix h, manipulate ε ✓

$$\frac{1}{\varepsilon}R(v) \to \frac{1}{\Phi_{\varepsilon,h}}R(v), \quad \Phi_{\varepsilon,h} := \max(\varepsilon, h^\theta)$$

Theorem (Entropy-inequality)

Assume that $\psi \in W_0^{1,\infty}$*, and* $\theta < \frac{1}{2}$ *we have*

$$\langle \partial_t \eta(u_h^\varepsilon) + \nabla \cdot \boldsymbol{q}(u_h^\varepsilon) - \frac{1}{\Phi_{\varepsilon,h}}\eta(u_h^\varepsilon)' R(u_h^\varepsilon), \psi\rangle \leq$$

$$C\Big(h^{1-2\theta}\|u_h^\varepsilon\|_{L^1(L^1)} + \big(h^{2-2\theta} + h^{1-\theta}\big)\|\nabla u_h^\varepsilon\|_{L^1(L^1)}\Big)$$

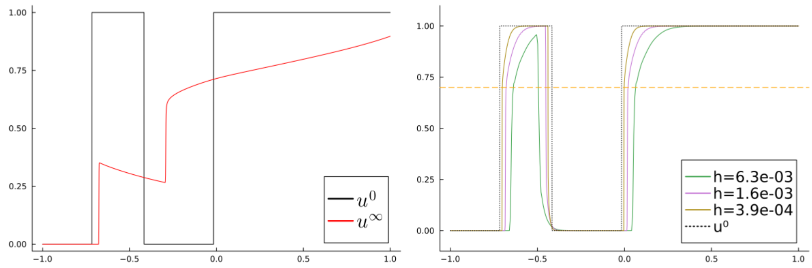
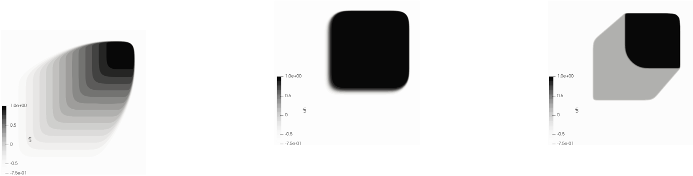


Figure: $\boldsymbol{f}(v) = \frac{1}{2\pi}\sin(2\pi v)$, $R(v) = v(1-v)(v-0.7)$, $\varepsilon = 10^{-8}$

Test, $\theta = 0.4$ (2D)



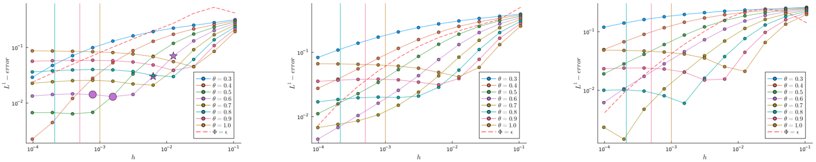
(a) $R(v) = 0$

(b) $R(v) = (v + \frac{3}{4})(1 - v)(v - \frac{1}{8})$

(c) $R(v) = (v + \frac{3}{4})(1 - v)(v - \frac{1}{8})(v + \frac{5}{8})(v - \frac{9}{16})$

Figure: $\boldsymbol{f}(v) = (\frac{1}{2}v^2, \frac{1}{2}v^2)$, $\varepsilon = 10^{-8}$, $h = 5 \times 10^{-3}$

Optimal parameters



(a) $\boldsymbol{f}(v) = v$ (b) $\boldsymbol{f}(v) = \frac{1}{2}v^2$ (c) $\boldsymbol{f}(v) = \frac{1}{2\pi}\sin(2\pi v)$

Figure: $R(v) = v(1-v)(v-0.7)$, $\varepsilon = 10^{-3}$

minimizing error $\Rightarrow$ lower envelope $\Rightarrow$

$$\theta \approx a + b(\log h)^{-1} \Rightarrow \quad \Phi_{\varepsilon,h} := \max(\varepsilon, \gamma h^{\theta}), \quad \theta \approx a, \ \gamma \approx e^{b}$$

Thank you :)