Bound-preserving & mass-conservative high-order methods for fourth-order elliptic problems and nonlinear parabolic equations in biology

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A phase-field model for phase separation [Cahn & Hilliard, J. Chem. Phys., 1958]:

$$\partial_t u = \nabla \cdot \big(M(u)\nabla w\big), \qquad w = -\Delta u + f(u), \qquad f = F'.$$

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$$-1 < u(x,t) < 1.$$

Motivation: Cahn–Hilliard equation

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- Energy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u) = -\int_D M(u)|\nabla w|^2 \leq 0, \qquad E(u) = \frac{1}{2}\|\nabla u\|^2 + \int_D F(u).$$

A fourth-order model for a thin liquid film [Bertozzi & Pugh, CPAM, 1996; Zhornitskaya & Bertozzi, SINUM, 2000]:

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A nonlinear diffusion equation with density-dependent diffusivity [Vázquez, *The Porous Medium Equation*, Oxford University Press, 2007]:

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Bound-preservation	$a \leq U_i^n \leq b,$
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Energy dissipation	$E(u_h^{n+1}) \leq E(u_h^n).$

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1. Stationary linear fourth-order elliptic problems
2. Nonlinear parabolic problems

- Discrete maximum principles

  Low order, mesh restrictions.

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  High-order bound-preservation. Mass-conservation is not automatic.

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- Bound-preservation + mass conservation

  Enriched Galerkin methods, optimization, redistribution.

  Barrenechea, Lederer & Rupp, 2025. Liu et al., SIAM J. Sci. Comput., 2024.

  Guermond & Wang, J. Comput. Phys., 2025.

- Energy stability

  Convex splitting, IEQ, SAV and Lagrange multiplier.

  Elliott & Stuart, SINUM, 1993. Shen & Yang, DCDS, 2010.

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- Bound-preservation and energy stability

  SAV and maximum-principle methods.

  Huang & Shen, SISC, 2021. Cheng, Wang & Zhao, preprint, 2025.

Fourth-order elliptic problems

High-order finite elements with
bound-preservation and mass-conservation

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High-order finite elements with
bound-preservation and mass-conservation

Time-dependent problems

High-order BDF-FEM schemes with
bound-preservation and mass-conservation

SAV stabilization for improved energy behavior

Preprint:
https://arxiv.org/abs/2605.23486
https://hal.science/hal-05630475v1

Fourth-order elliptic problems

Let $D \subset \mathbb{R}^d$, $d \in \{1,2,3\}$. Find (u,w) such that

$$\begin{cases} u = \nabla \cdot (M \nabla w) + f_1, \\ w = -\nabla \cdot (\kappa \nabla u) + f_2, \end{cases} \qquad \text{in } D,$$

with homogeneous Neumann boundary conditions.

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with homogeneous Neumann boundary conditions.

Assumptions

$$0 < M_1 \leq M \leq M_2, \qquad 0 < \kappa_1 \leq \kappa \leq \kappa_2,$$

$$f_1, f_2 \in L^2(D), \qquad \frac{1}{|D|} \int_D f_1 \in [a, b], \qquad \int_D f_2 = 0.$$

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$$u(x) \in [a,b], \qquad \int_D u = \int_D f_1.$$

Define

$$\mathcal{K} := \{v \in H^1(D) : a \leq v \leq b \text{ a.e.}\}.$$

Find $(u, w) \in \mathcal{K} \times H^1(D)$ such that

$$\begin{cases} (u, v) + (M\nabla w, \nabla v) = (f_1, v) \quad \forall v \in H^1(D), \\ (w, \xi - u) \leq (\kappa \nabla u, \nabla(\xi - u)) + (f_2, \xi - u) \quad \forall \xi \in \mathcal{K}. \end{cases}$$

Variational inequality

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The constraint $u \in [a, b]$ is built into $\mathcal{K}$.
The mass-conservation is built by taking $v = 1$.

See [Han ,2024, Ch. 3]; [Blank, Butz & Garcke, 2011].

Let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular mesh of D, and

$$V_h^p := \operatorname{span}\{\phi_i\}_{i\in\mathcal{V}}, \qquad \mathcal{V} := \{1{:}I\}.$$

$$V_h^{p,+} := \left\{ v_h = \sum_{i\in\mathcal{V}} V_i\phi_i \in V_h^p \;\middle|\; V_i \in [a,b],\ \forall i \in \mathcal{V} \right\}.$$

Find $(u_h, w_h) \in V_h^{p,+} \times V_h^p$ such that

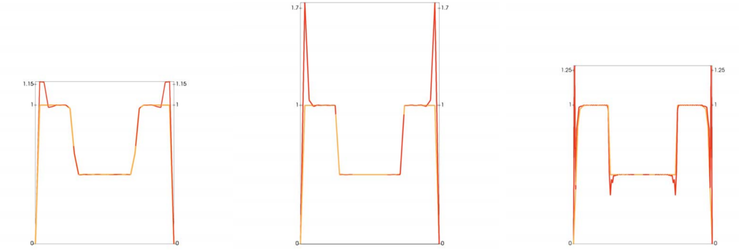
$$\begin{cases} (u_h, v_h) + (M\nabla w_h, \nabla v_h) = (f_1, v_h), & \forall v_h \in V_h^p, \\ (w_h, \xi_h - u_h) \le (\kappa\nabla u_h, \nabla(\xi_h - u_h)) + (f_2, \xi_h - u_h), & \forall \xi_h \in V_h^{p,+}. \end{cases}$$

No acute-angle condition [Barrenechea, John & Knobloch, 2024]
Solver: PDAS (semismooth Newton) [Hintermüller, Ito & Kunisch, 2002]

Related VI methods: [Blank, Butz & Garcke, 2011]; [Barrenechea, Georgoulis & Pryer, 2023]; [Kirby & Shapero, 2024]; [Dong, Ern & Wang, 2025].

Oscillation can be significantly alleviated

[Barrenechea, Georgoulis & Pryer, 2023]



(A) $\epsilon = 10^{-4}$, $k = 1$. (B) $\epsilon = 10^{-7}$, $k = 1$. (C) $\epsilon = 10^{-7}$, $k = 2$.

FIG. 7. A cross section taken about the $x = y$ plane of the finite element solution, coloured red presenting oscillations at the boundary layer, and the approximation given by (5.3), coloured green with no oscillations at the boundary. Panels (A) and (B) are computed using $k = 1$, while (C) depicts the approximations obtained using $k = 2$.

Define the admissible set

$$Y := \left\{ v_h \in V_h^{p,+} : \int_D v_h = \int_D f_1 \right\}.$$

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The component u_h is the unique minimizer of

$$\min_{v_h \in Y} \left\{ \frac{1}{2} \|\sqrt{\kappa} \nabla v_h\|^2 + (f_2, v_h) + \frac{1}{2} \|v_h - f_1\|_{H_h^{-1}}^2 \right\}.$$

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Consequences

- existence and uniqueness of u_h,
- bound-preservation,
- mass-conservation.

Here $\|g\|_{H_h^{-1}}^2 = (g, (-\Delta_h^M)^{-1} g)$.

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$$u \in C^0(\overline{D}; [a,b]) \cap W^{p,\infty}(D) \cap H^{p+1}(D), \qquad w \in H^{p+1}(D).$$

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$$\|\nabla(u - u_h)\| \leq Ch^p.$$

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Two ingredients:

1. Construct an optimal admissible candidate:

$$\Pi_h u \in Y, \qquad \|\nabla(\Pi_h u - u)\| \leq Ch^p.$$

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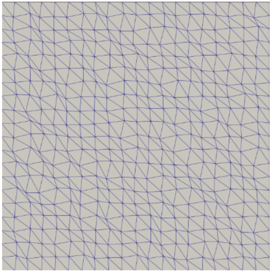
2. Apply standard FEM consistency and stability arguments to the H^1–H^{-1} representation of the scheme.

Consider

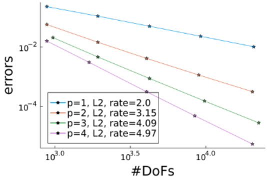
$$\begin{cases} u = \nabla \cdot ((1+x)\nabla w) + f_1, \\ w = -\Delta u, \end{cases} \qquad D = (0,1)^2,$$

with

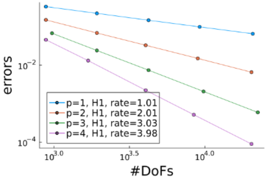
$$u(x,y) = \cos(4\pi x)\cos(4\pi y).$$



(a) Mesh with obtuse triangles.



(b) Convergence test, L^2-norm.

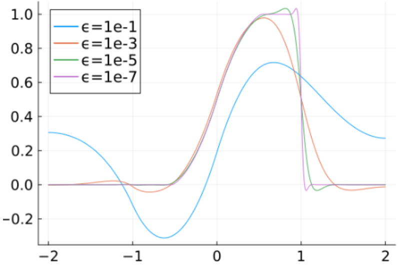


(c) Convergence test, H^1-norm.

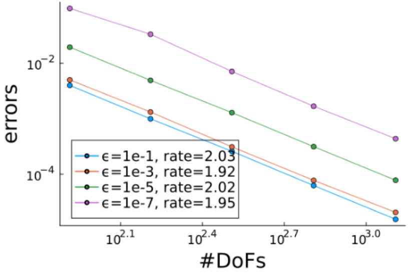
Consider

$$\begin{cases} u = \nabla \cdot (M \nabla w) + f_1, \\ w = -\Delta u + f_2, \end{cases} \qquad D = (-2, 2).$$

$$M(x) = \begin{cases} 2 & |x| < 0.5, \\ \varepsilon & \text{otherwise}, \end{cases} \quad f_1(x) = \begin{cases} 1 & 0 < x < 1, \\ 0 & \text{otherwise}, \end{cases} \quad f_2(x) = \begin{cases} 5 & -1 < x < 0, \\ 0 & \text{otherwise}. \end{cases}$$



(a) Reference solutions.



(b) Convergence rate in the L^2-norm, $p = 1$.

Application on parabolic problems

Consider [Bertozzi & Pugh, 1996; Zhornitskaya & Bertozzi, 2000]

$$\begin{cases} \partial_t u = \nabla \cdot \big(M(u)\nabla w\big), & \text{in } D \times (0,T], \\ w = -\Delta u, & \text{in } D \times (0,T], \\ \partial_{\boldsymbol{n}} u = \partial_{\boldsymbol{n}} w = 0, & \text{on } \partial D \times (0,T], \\ u(\cdot,0) = u_0. \end{cases}$$

For $k \in \{1, \ldots, 5\}$, find

$$(u_h^{n+1}, w_h^{n+1}) \in V_h^{p,+} \times V_h^p$$

such that

$$\begin{cases} \left(\dfrac{\alpha_k u_h^{n+1} - A_k(u_h^n)}{\tau}, v_h \right) + \big(M(\overline{u}_h^{n+1}) \nabla w_h^{n+1}, \nabla v_h \big) = 0 & \forall v_h \in V_h^p \\ (w_h^{n+1}, \xi_h - u_h^{n+1}) \leq (\nabla u_h^{n+1}, \nabla(\xi_h - u_h^{n+1})) & \forall \xi_h \in V_h^{p,+} \end{cases}$$

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Here (α_k, A_k) denotes BDFk, and

$$\overline{u}_h^{n+1} := \sum_{i \in \mathcal{V}} \phi_i \min\{b, \max\{a, B_k(U_i^n)\}\},$$

where B_k is the standard extrapolation operator.

Explicit mobility, implicit variational inequality.

Properties of the scheme

For every time step, a solution (u_h^{n+1}, w_h^{n+1}) exists, and u_h^{n+1} is unique.

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In fact,

$$u_h^{n+1} = \operatorname*{arg\,min}_{v \in Y^{n+1}} \left\{ \frac{\alpha_k}{2} \|\nabla v\|^2 + \frac{1}{2\tau} \|\alpha_k v - A_k(u_h^n)\|^2_{H_h^{-1,n+1}} \right\},$$

where

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For BDF1,

$$\frac{1}{2}\|\nabla u_h^{n+1}\|^2 \leq \frac{1}{2}\|\nabla u_h^n\|^2.$$

Energy stability is proved for BDF1 only.

Singular lubrication test

Consider [Cheng & Shen, 2022; Zhornitskaya & Bertozzi, 2000]

$$\begin{cases} \partial_t u = \nabla \cdot \left(\sqrt{u}\, \nabla w\right), \\ w = -\Delta u, \end{cases} \qquad D = (-1, 1),$$

with $u_0(x) = 0.8 - \cos(\pi x) + 0.25 \cos(2\pi x)$.
The solution develops a temporal singularity near $t_\star \approx 7.4 \times 10^{-4}$.

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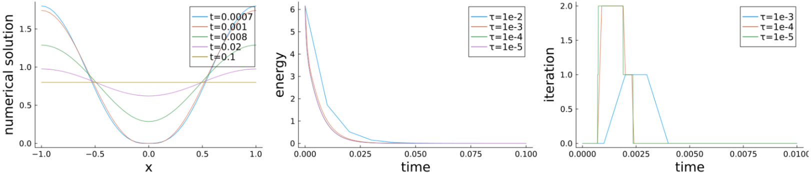
$$50 \text{ nodes}, \qquad p = k = 1.$$



(a) Numerical solution, $\tau = 10^{-4}$. (b) Energy evolution for different τ. (c) Iteration number for different τ.

Stable through the singularity without mobility regularization.

Consider [Cahn & Hilliard, 1958]

$$\begin{cases} \partial_t u = \nabla \cdot \big(M(u)\nabla w\big), & \text{in } D \times (0,T], \\ w = -\Delta u + f(u), & \text{in } D \times (0,T], \\ \partial_{\boldsymbol{n}} u = \partial_{\boldsymbol{n}} w = 0, & \text{on } \partial D \times (0,T], \\ u(\cdot, 0) = u_0, \end{cases}$$

Introduce [Shen, Xu & Yang, 2019]

$$r(t) := \sqrt{E_1(u(t))}.$$

For $k \in \{1, \dots, 5\}$, find $(u_h^{n+1}, r^{n+1}, w_h^{n+1}) \in V_h^{p,+} \times \mathbb{R} \times V_h^p$ such that

$$\begin{cases} \left(\dfrac{\alpha_k u_h^{n+1} - A_k(u_h^n)}{\tau}, v_h\right) + \left(M(\overline{u}_h^{n+1})\nabla w_h^{n+1}, \nabla v_h\right) = 0 & \forall v_h \in V_h^p \\ (w_h^{n+1}, \xi_h - u_h^{n+1}) \leq (\nabla u_h^{n+1}, \nabla(\xi_h - u_h^{n+1})) & \\ \qquad + \left(\dfrac{r^{n+1} f(\overline{u}_h^{n+1})}{\sqrt{E_1(\overline{u}_h^{n+1})}}, \xi_h - u_h^{n+1}\right) & \forall \xi_h \in V_h^{p,+} \\ \alpha_k r^{n+1} - A_k(r^n) = \left(\dfrac{f(\overline{u}_h^{n+1})}{2\sqrt{E_1(\overline{u}_h^{n+1})}}, \alpha_k u_h^{n+1} - A_k(u_h^n)\right) & \end{cases}$$

Here $\overline{u}_h^{n+1}$ is the clipped extrapolation defined for the lubrication scheme.

Explicit nonlinear terms and one scalar auxiliary variable.

Properties of the scheme

For every time step, the scheme admits a solution, and u_h^{n+1} is unique.

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$$\min_{(v,s)\in X^{n+1}} \left\{ \frac{\alpha_k}{2}\|\nabla v\|^2 + \alpha_k s^2 + \frac{1}{2\tau}\|\alpha_k v - A_k(u_h^n)\|^2_{H_h^{-1,n+1}} \right\}.$$

Therefore,

$$U_i^{n+1} \in [a,b] \qquad \forall i \in \mathcal{V},$$

and

$$\int_D u_h^{n+1} = \int_D u_h^n.$$

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For BDF1,

$$\frac{1}{2}\|\nabla u_h^{n+1}\|^2 + (r^{n+1})^2 \leq \frac{1}{2}\|\nabla u_h^n\|^2 + (r^n)^2.$$

Modified energy stability is proved for BDF1 only.

Consider [Cheng & Shen, 2022]

$$\begin{cases} \partial_t u = \nabla \cdot \big((1-u^2)\nabla w\big), \\ w = -0.01\Delta u + \log(1+u) - \log(1-u) - 5u, \end{cases} \qquad D = (0,1)^2,$$

with

$$u_0(x,y) = 0.2 + 0.05\,\mathrm{rand}(-1,1), \qquad T = 1.5 \times 10^{-1}.$$

The solution must remain in $(-1,1)$, $M(u) = 1-u^2 \longrightarrow 0$ as $u \longrightarrow \pm 1$.

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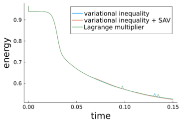
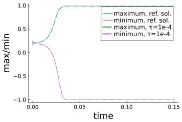
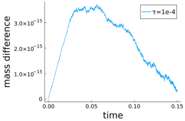
$$C_0 = 1, \qquad p = 1, \qquad k = 2, \qquad 50 \times 50 \text{ nodes.}$$



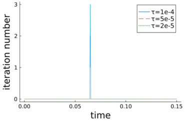
(a) Comparison of energy evolution, $\tau = 10^{-4}$.



(b) Maximum and minimum of u_h^n.



(c) Mass-conservation of u_h^n.



(d) Comparison of iteration number.

Consider

$$\begin{cases} \partial_t u = \nabla \cdot \big(K(u)\nabla u\big), & \text{in } D \times (0,T], \\ \partial_{\boldsymbol{n}} u = 0, & \text{on } \partial D \times (0,T], \\ u(\cdot,0) = u_0. \end{cases}$$

Introduce the fourth-order regularization

$$\begin{cases} \partial_t u = \nabla \cdot \big(K(u)\nabla u\big) + \sqrt{\varepsilon}\,\Delta w, \\ w = -\sqrt{\varepsilon}\,\Delta u. \end{cases}$$

The perturbation error is formally $\mathcal{O}(\varepsilon)$.

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Set

$$\varepsilon = h^{p+1}.$$

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Well-posedness is not covered by the current analysis.

Numerical experiments indicate stable solvability.

Consider

$$\partial_t u = \nabla \cdot \big((1+u)\nabla u\big) + f_1, \qquad D = (0, 2\pi)^2,$$

with $u(x,y,t) = \big(\cos(x)\cos(y) - \frac{3}{8\pi}x^4 + x^3\big)\cos(t)$, $T = 1$.

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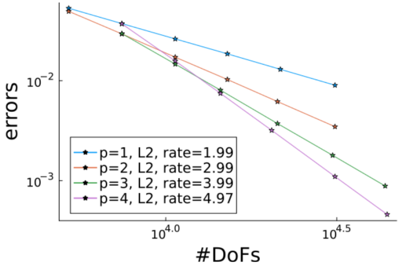
$$\tau = \frac{h}{2}, \qquad k = p+1, \qquad \partial_{\boldsymbol{n}} w_h = 0 \text{ although } \partial_{\boldsymbol{n}} w \neq 0.$$



Optimal L^2-rates for $p = 1, \ldots, 4$.

Consider [Cheng & Shen, 2022; Vázquez, 2007]

$$\partial_t u = m\nabla \cdot \big(u^{m-1}\nabla u\big), \qquad D = (-5,5), \qquad T = 1.2,$$

The Barenblatt solution is

$$u(x,t) = \frac{1}{t_0^\alpha}\left[\max\left(0,\ 1-\alpha\frac{m-1}{2m}\frac{x^2}{t_0^{2\alpha}}\right)\right]^{\frac{1}{m-1}}, \quad t_0 = t+1, \quad \alpha = \frac{1}{m+1}.$$

The solution has compact support, moving interface, and $u \in H^s(D)$ with $s = 2, 1/2 + 1/(m-1)$ for $m = 2$ and $m \geq 3$, respectively.

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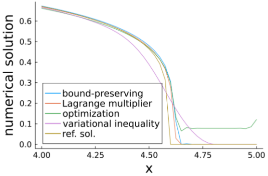
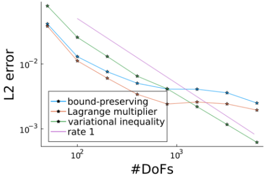
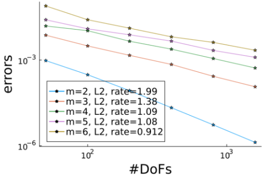
$$p = 1, \qquad k = 2, \qquad \tau = \frac{h}{10}.$$



(a) Numerical solution produced by different methods, $m = 6$.



(b) Convergence test for different methods, $m = 6$.



(c) Convergence test for the proposed method.

Linear fourth-order elliptic problems

high-order FEM $\quad + \quad$ variational inequality

bound-preservation, mass-conservation, $\|\nabla(u - u_h)\| \leq Ch^p.$

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BDF $\quad + \quad$ variational inequality $\quad + \quad$ SAV stabilization

bound-preservation, mass-conservation, energy stability for $k = 1.$

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Nonlinear second-order parabolic problems

fourth-order regularization $\quad + \quad$ BDF $\quad + \quad$ variational inequality

bound-preservation and mass-conservation.

1. **Time-dependent problems**

well-posedness and error estimates

Future work

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Thank you for your attention!

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`https://wangzuodong1997.github.io/`

Submitted manuscript:

`https://arxiv.org/abs/2605.23486`

`https://hal.science/hal-05630475v1`