Efficient numerical schemes for evolution equations with singularities and shocks

PhD defense

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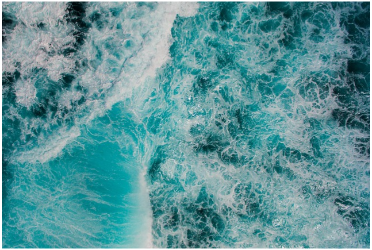
- $\partial_t u + \nabla \cdot f(u) = R(u)$
  Hyperbolic conservation laws with source term
- $\partial_{tt} u - \Delta u = f$
  Linear acoustic wave equations



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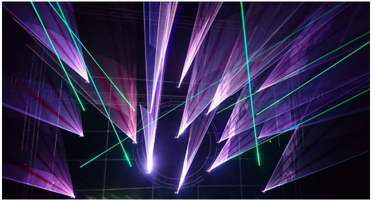
Shocks, interfaces

- $\Omega\cdot\nabla_{\boldsymbol{x}}\Psi+\sigma^t\Psi=\frac{\sigma^s}{4\pi}\int_{S^2}\Psi+q$
  Neutron transport equations
  $\Big(3(\boldsymbol{x})+2(\Omega)\Big)d$



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Highly-contrasted coefficients

- $\partial_t u - \Delta u + \varepsilon^{-2}(u^3 - u) = 0$
  Allen–Cahn equation

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Sharp interfaces

- Physically meaningful numerical solution:
  bound-preserving, energy-decay, entropy-decay, ...

- Well-captured shocks and singularities:
  correct wave speed, as small as possible oscillation, ...

Outline

1. Conservation laws with stiff reaction:
   [Ern, Guermond, Wang, 24, J. Sci. Comput.]

2. Neutron transport equation:
   [Guermond, Wang, 25, J. Comput. Phys.]

3. Allen–Cahn equation:
   [Dong, Ern, Wang, accepted, Comput. Math. Appl.]

4. Linear acoustic wave equation:
   [Dong, Mascotto, Wang, submitted, Numer. Math.]
   [Dong, Georgoulis, Mascotto, Wang, submitted, Numer. Math.]

01

Conservation laws with stiff reaction

[Ern, Guermond, Wang, 24, J. Sci. Comput.]

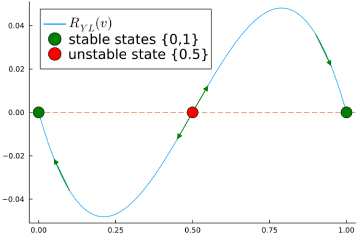
$$\partial_t u^\varepsilon + \nabla \cdot \boldsymbol{f}(u^\varepsilon) = \frac{1}{\varepsilon} R(u^\varepsilon) \qquad \text{in } D \times (0,T]$$

with u_0 and suitable BC, $\varepsilon > 0$

Examples for $\boldsymbol{f}$

- Linear:
  $f(v) = v$
- Burgers:
  $f(v) = \frac{1}{2}v^2$
- KPP:
  $\boldsymbol{f}(v) = (\sin v, \cos v)$

Example for R



$$R_{YL}(v) = v(1-v)(v-\tfrac{1}{2})$$

Example [LeVeque, Yee, 90]

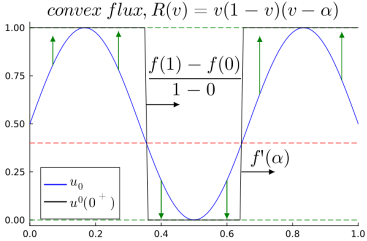
$$\partial_t u^\varepsilon + \partial_x u^\varepsilon = \tfrac{1}{\varepsilon} R_{YL}(u^\varepsilon)$$

We focus on $R(v) = v(1-v)(1-\alpha)$, $\alpha \in (0,1)$, for simplicity

- Invariant-domain-preserving (IDP):
  $u_0(\boldsymbol{x}) \in [0,1] \Longrightarrow u^\varepsilon(\boldsymbol{x},t) \in [0,1]$

- Entropy-inequality:
  $\partial_t \eta(u^\varepsilon) + \nabla \cdot \boldsymbol{q}(u^\varepsilon) \le \frac{1}{\varepsilon}\eta'(u^\varepsilon)R(u^\varepsilon)$,
  η is convex, $\boldsymbol{q}(v) := \int_0^v \eta'(s)\boldsymbol{f}'(s)\mathrm{d}s$

- There exists a unique u^0 (at least in 1D) [Fan, Jin, Teng, 00]
- $u^\varepsilon \xrightarrow{\varepsilon} u^0$ exponentially fast: reaction on characteristic line
- u^0 is piecewise constant, what are the shock speeds?



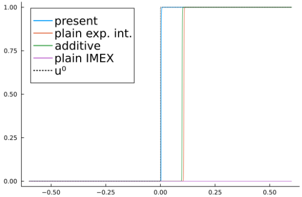
$$f'(\alpha) \neq \frac{f(1)-f(0)}{1-0}$$

- In general, u^0 may not satisfy Rankin–Hugoniot relation
- Approximating u^0 is challenging

In the regime $\varepsilon \ll h$:

- Capture shocks with correct wave speed
- IDP: $u_h^\varepsilon(\boldsymbol{x}, t) \in [0, 1]$
- Asymptotic-preserving (AP): $u_h^\varepsilon \approx u_h^0$
- Entropy inequality:
$\partial_t \eta(u_h^\varepsilon) + \nabla \cdot \boldsymbol{q}(u_h^\varepsilon) - \frac{1}{\varepsilon}\eta(u_h^\varepsilon)' R(u_h^\varepsilon) \leq \Lambda(h) \to 0$

- Classical schemes are not AP.



$$f(v) = v$$
$$u_0(x) = \begin{cases} 0 & x < -0.5 \\ 1 & x \geq -0.5 \end{cases}$$
$$\varepsilon = 10^{-3},\ h = 3.9 \times 10^{-4}$$
$$R(v) = v(1-v)(v-0.9)$$

- Some partially successful results:
  - [Bao, Jin, 02]: AP for 1D, convex flux, shock-type IC
  - [Svärd, Mishra, 11]: AP for 1D, convex flux; no interaction between rarefaction and shock

Our contribution

IDP-AP scheme for arbitrary d, $\boldsymbol{f}$ and u_0

$$u_h^{n,\varepsilon} \xrightarrow{\text{transport}} w_h^{n+1,\varepsilon} \xrightarrow{\text{reaction}} u_h^{n+1,\varepsilon}$$

- Transport ($\frac{w_h^{n+1,\varepsilon} - u_h^{n,\varepsilon}}{\tau} + \nabla \cdot \boldsymbol{f}(u_h^{n,\varepsilon}) = 0$):
  Stabilized FEM, upwind, Lax-Friedrichs, Godunov, ...
  e.g., [Eymard, Gallouet, Herbin, 00], [Guermond, Nazarov, 14]

- Reaction ($u_h^{n+1,\varepsilon} = v(\tau), d_t v = \frac{1}{\varepsilon} R(v),\ v(0) = w_h^{n+1,\varepsilon}$):
  - > Original ODE is expensive to solve
  - > Fast IDP update: $d_t v = \frac{1}{\varepsilon} v(1-v)(w_h^{n+1,\varepsilon} - \alpha)$
    $\Rightarrow u_h^{n+1,\varepsilon} = \frac{w_h^{n+1,\varepsilon} \exp((\tau/\varepsilon)(w_h^{n+1,\varepsilon} - \alpha))}{1 + w_h^{n+1,\varepsilon}\left(\exp((\tau/\varepsilon)(w_h^{n+1,\varepsilon} - \alpha)) - 1\right)}$

- $u_h^{n,\varepsilon}(\boldsymbol{x}) \in [0,1] \Rightarrow w_h^{n+1,\varepsilon}(\boldsymbol{x}) \in [0,1] \Rightarrow u_h^{n+1,\varepsilon}(\boldsymbol{x}) \in [0,1]$

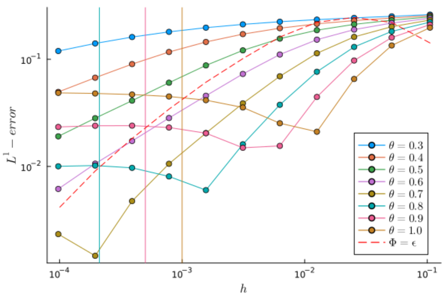
$$\frac{1}{\varepsilon}R(v) \to \frac{1}{\Phi_{\varepsilon,h}}R(v), \quad \Phi_{\varepsilon,h} := \max(\varepsilon, h^\theta)$$

Theorem (Entropy inequalities)

Assume that $\eta \in \mathcal{C}^2$, $\varepsilon \ll h^\theta$ *and* $\psi \in W_0^{1,\infty}$, *we have*

$$\langle \partial_t \eta(u_h^\varepsilon) + \nabla \cdot \boldsymbol{q}(u_h^\varepsilon) - \frac{1}{\Phi_{\varepsilon,h}}\eta(u_h^\varepsilon)' R(u_h^\varepsilon), \psi \rangle \leq$$
$$C\Big(h^{1-2\theta} \|u_h^\varepsilon\|_{L^1(L^1)} + h^{1-\theta}\|\nabla u_h^\varepsilon\|_{L^1(L^1)} \Big)$$

Question: optimal θ to minimize error?



$R(v) = v(1-v)(v-0.7)$
$\boldsymbol{f}(v) = \frac{\sin(2\pi v)}{2\pi}$, $\varepsilon = 10^{-3}$
•: $R(\cdot)/h^\theta$, $\theta \in \{0.3, \ldots, 1\}$
dashed line: $R(\cdot)/\varepsilon$
colored vertical lines: $h^\theta = \varepsilon$

minimizing error $\Rightarrow$ lower envelope $\Rightarrow$

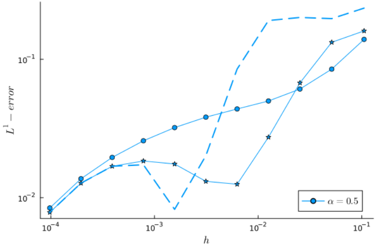
$$\theta \approx a + b(\log h)^{-1} \Rightarrow \quad \Phi_{\varepsilon,h} := \max(\varepsilon, \gamma h^\theta), \quad \theta \approx a, \ \gamma \approx e^b$$

- (θ, γ) depends on model parameters

- Still possible to consider all-purpose parameters

$$(\theta, \gamma) := \begin{cases} (0.1, 0.05) & \text{linear flux} \\ (0.4, 0.1) & \text{nonlinear flux} \end{cases}$$

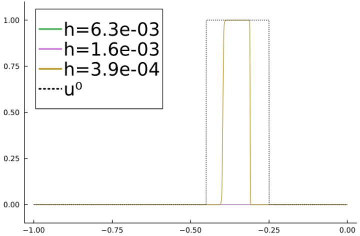
- $f(v) = \frac{\sin(2\pi v)}{2\pi}$, $R(v) = v(1-v)(v-0.5)$, $\varepsilon = 10^{-3}$



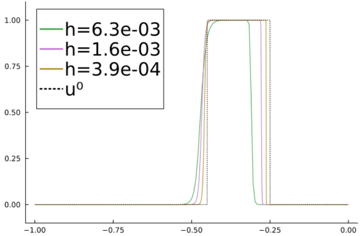
(θ, γ): optimal values $\star$, all-purpose values $\bullet$; dashed line $\Phi_{\varepsilon,h} = \varepsilon$.

- Similar result for other $\boldsymbol{f}$ and R

- $\boldsymbol{f}(v) = \frac{1}{2}v^2$, $R(v) = v(1-v)(v-0.9)$, $\varepsilon = 10^{-3}$
- Interaction between shock and rarefaction



(a) [Svärd, Mishra, 11]



(b) our scheme

- Shocks incorrectly approximated in (a); correctly approximated in (b)

- An IDP-AP scheme is proposed, based on operator-splitting and cut-off techniques
- Optimal cut-off parameters are numerically investigated
- High-order schemes are a future research direction

02

Neutron transport equation

[Guermond, Wang, 25, J. Comput. Phys.]

Find $\Psi : D \times S^2 \to \mathbb{R}^+$ s.t.

$$\Omega \cdot \nabla_{\boldsymbol{x}} \Psi + \sigma^t \Psi = \frac{\sigma^s}{4\pi} \int_{S^2} \Psi + q$$

with non-negative inflow BC,
total and scattering cross sections $\sigma^t \geq \sigma^s \geq 0$ (can vary on D)

- Key properties:
  - Positivity-preserving: $\Psi(\boldsymbol{x}, \Omega) \geq 0$
  - $|\sigma^s| \to \infty \Rightarrow$ diffusion limit Ψ^0

- S_N for angular discretization
- Main challenge: find a method with
  ➤ positivity-preserving ➤ AP ➤ high-order
  ➤ oscillation-free ➤ conservative ➤ fast post-processing

	$\Psi \geq 0$	AP	high-order	no osc.	conserv.	fast post-proc.
dG(0) + modif.	✓	✓	✗	✓	✓	-
dG(p)	✗	✓	✓	✗	✓	-
dG(p) + optim.	✓	✓	✓	✓	✓	✗
cG	✗	✓	✓	✗	✓	-
cG + $h\Delta\Psi$ + rescaling	✓	✓	✗	✓	✗	-

References: [Chandrasekhar, 50], [Larsen, Morel, Miller, 87], [Gosse, Toscani, 02], [Guermond, Kanschat, 10], [Buet, Després, Frank, 12], [Guermond, Popov, Ragusa, 20], [Yee, Olivier, Haut, Holec, Tomov, Maginot, 20]

Novel limiting process

➤ positivity-preserving ➤ AP ➤ high-order
➤ oscillation-free ➤ conservative ➤ fast (linear complexity)

A two-step post-process:

1. Local limiting: temper oscillation
   - loop on all dofs:
   - apply local limiter based on mass redistribution
2. Global limiting: impose global a priori bounds
   - apply global limiter based on cut-off technique

$$\Omega \cdot \nabla_{\boldsymbol{x}} u + \sigma u = q \qquad \text{in } D \subset \mathbb{R}^d$$

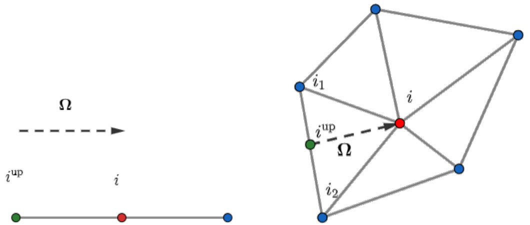
with suitable BC, fixed $\Omega \in \mathbb{R}^d$, $d = 1, 2, 3$, $\sigma, q \geq 0$

Solution method:

- stabilized high-order numerical method (e.g., edge stabilization/CIP [Burman, Hansbo, 04]) $\Rightarrow \{u_i\}_{i \in \mathcal{V}}$;
- local limiting to temper small oscillation (repeat a few times):
  - > estimate local bounds $\Rightarrow \{(u_i^{\max}, u_i^{\min})\}_{i \in \mathcal{V}}$
  - > apply local limiter $\Rightarrow \{\tilde{u}_i\}_{i \in \mathcal{V}}$
- Global limiting to guarantee positivity $\Rightarrow \{u_i^+\}_{i \in \mathcal{V}}$

Based on method of characteristics [Lathrop, 69];
$\sigma = const$ for simplicity

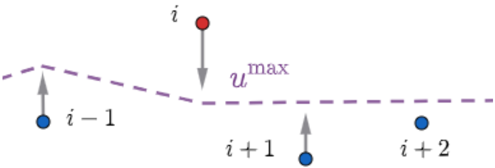
$$\begin{cases} \Omega \cdot \nabla_{\boldsymbol{x}} u + \sigma u = q \\ u|_{\partial D^-} = u^{\mathsf{up}} \end{cases} \Leftrightarrow u(x) = u^{\mathsf{up}} e^{\frac{\sigma}{|\Omega|}|x^{\mathsf{up}} - x|} + \int_{x^{\mathsf{up}}}^{x} \frac{q}{|\Omega|} e^{\frac{\sigma}{|\Omega|} s} \, \mathrm{d}s$$



- Upwinding node i^{up} defined using local dof stencil $\mathcal{I}(i)$
- Maximizing/minimizing $q \Rightarrow$ local bound

Main steps (repeat a few times):

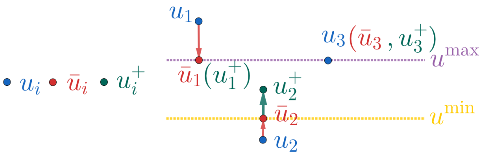
- loop on all dofs $i \in \mathcal{V}$:
  1. compute $\{u_i^{\max}, u_i^{\min}\}$
  2. apply local limiter on each dof $i \in \mathcal{V}$
     - ⋆ $u_i > u_i^{\max} \Rightarrow$ decrease u_i, increase $\{u_j\}_{j \in \mathcal{I}(i)}$
     - ⋆ $u_i < u_i^{\min} \Rightarrow$ increase u_i, decrease $\{u_j\}_{j \in \mathcal{I}(i)}$



- Locally mass conservative
- Converging in a few number of iterations on all dofs

Main steps:

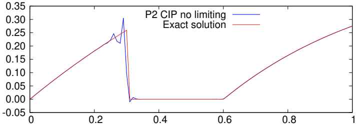
- set lower/upper bound to $u^{\min}$, $u^{\max}$;
- cut-off $\Rightarrow \{\bar{u}_i\}_{i\in\mathcal{V}}$;
- small modification on dofs based on $\sum_{i\in\mathcal{V}} m_i u_i \Rightarrow \{u_i^+\}_{i\in\mathcal{V}}$.



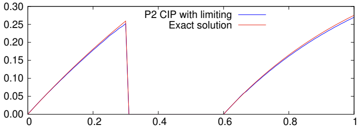
Lemma (properties of global limiter)

$$u^{\min} \le u_i^+ \le u^{\max}, \qquad \sum_{i\in\mathcal{V}} m_i u_i^+ = \sum_{i\in\mathcal{V}} m_i u_i$$

- $\Omega = 1$, $D = (0, 1)$, $u(0) = 0$, $q = O(1)$
  highly-contrasted σ between 1 and 10^3
- Simulation with $h \approx 2 \times 10^{-2}$, $\mathbb{P}_2$



(a) Galerkin + CIP



(b) Galerkin + CIP + limiting

- $\Omega = (1,0)$, $D = (0,1)^2$, $q = O(1)$
  highly-contrasted σ between 1 and 10^3

$\mathbb{P}_1$			$\mathbb{P}_2$			$\mathbb{P}_3$		
I	L^1-Err	rate	I	L^1-Err	rate	I	L^1-Err	rate
961	5.08E-02	–	1681	2.32E-02	–	961	4.12E-02	–
3721	2.34E-02	1.15	6561	1.04E-02	1.18	3721	1.96E-02	1.10
14641	9.62E-03	1.30	25921	3.74E-03	1.49	14641	7.08E-03	1.49
58081	3.18E-03	1.61	103041	1.09E-03	1.79	58081	1.71E-03	2.06
231361	8.28E-04	1.95	410881	4.64E-04	1.23	231361	4.18E-04	2.04

$$\Omega \cdot \nabla_{\boldsymbol{x}} \Psi + \sigma^t \Psi = \frac{\sigma^s}{4\pi} \int_{S^2} \Psi + q$$

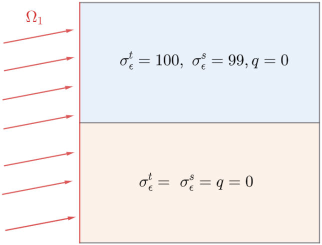
Main steps:

1. Discretization:
   - angular: discrete ordinate/S_N:
     quadrature $\{\Omega_k, \mu_k\}_{k \in K}$ on unit sphere
   - space: Galerkin + CIP (numerically AP)
   - numerical solution: $\{\Psi_{k,h}\}_{k \in K}$, dofs $\{\Psi_{k,i}\}_{k \in K, i \in \mathcal{V}}$
2. Source iteration (on n):
   - $\Omega_k \cdot \nabla_{\boldsymbol{x}} \Psi_{k,h}^{n+1} + \sigma^t \Psi_{k,h}^{n+1} = \frac{\sigma^s}{4\pi} \sum_l \mu_l \Psi_{l,h}^n + q := q^n$
   - local bounds estimator $\Rightarrow \{(\Psi_{k,i}^{\max}, \Psi_{k,i}^{\min})\}_{k,i}$ using q^n
     local limiting $\Rightarrow \{\tilde{\Psi}_{k,i}^{n+1}\}_{k,i}$
   - global limiting with $(\Psi^{\min}, \Psi^{\max}) = (0, +\infty) \Rightarrow \{\Psi_{k,i}^{n+1,+}\}_{k,i}$

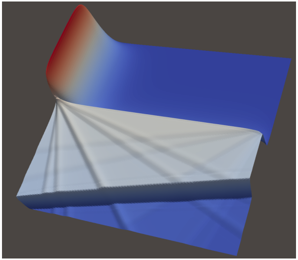
- $\sigma^t = \sigma^s = \varepsilon^{-1}$, $q = 2\varepsilon\pi^2 \sin(\pi x)\sin(\pi y)/3$, zero BC, $\psi^0 = \lim_{\varepsilon\to 0} \Psi = \sin(\pi x)\sin(\pi y)$
- $\mathbb{P}_1$, $h \gg \varepsilon$, $\bar{\Psi}_h$ as average of $\Psi_{k,h}$ over k
- convergence $\|\bar{\Psi}_h - \psi^0\|$ in L^2- and H^1-norms

ε	h	L^2-error	rate	H^1-error	rate
1e-6	1e-1	1.23e-2	-	1.95e-2	-
	5e-2	3.14e-3	2.12	5.75e-3	1.90
	2.5e-2	7.84e-4	2.08	1.98e-3	1.60
	1.25e-2	1.92e-4	2.06	7.06e-4	1.51
	6.25e-3	4.59e-5	2.08	2.35e-4	1.60

- $\Psi(\boldsymbol{x}, \Omega) = \begin{cases} 1 & \text{if } \boldsymbol{n}(\boldsymbol{x}) \cdot \Omega < 0 \text{ and } \Omega = \Omega_1 \\ 0 & \text{otherwise} \end{cases}$
- $h \approx 2.6 \times 10^{-3}$, $\max_{\boldsymbol{x} \in D} \sigma^s \approx 10^2$, S_6 (few angulars)



(a) geometry & parameters



(b) $\mathbb{P}_1$, $\bar{\Psi}_h$

- A fast, mass conservative and bound-preserving post-processing is proposed and numerically tested.

- Possible improvement: recall source iteration

$$\Omega_k \cdot \nabla_{\boldsymbol{x}} \Phi_{h,k}^{n+1} + \sigma^t \Phi_{h,k}^{n+1} = \frac{\sigma^s}{4\pi} \sum_l \mu_l \Psi_{l,h}^n + q$$

  - small oscillation near discontinuity
  - RHS/inflow dof are not always positive
  - local bound is not always positive/accurate
  - need more reliable local bound

03

Allen–Cahn equation

[Dong, Ern, Wang, accepted, Comput. Math. Appl.]

$$\partial_t u - \Delta u + \frac{1}{\varepsilon^2} f(u) = 0 \qquad \text{in } D \times (0, T)$$

with $u_0 \in H^1(D)$, $\partial_{\boldsymbol{n}} u|_{\partial D} = 0$, $\varepsilon > 0$.

- $F(v) := \frac{1}{4}(v^2 - 1)^2$, $f(v) := F'(v) = v(v^2 - 1)$ with stable states ± 1
- Energy decay: $\mathcal{J}_\varepsilon(v) := \frac{1}{2}\|\nabla v\|^2 + \frac{1}{\varepsilon^2}\int_D F(v)$ $\sup_{t \in J} \mathcal{J}_\varepsilon(u(t)) + \int_0^T \|\partial_t u(t)\|^2 \mathrm{d}t \leq \mathcal{J}_\varepsilon(u_0)$

- A narrow transition region of $O(\varepsilon)$ thickness exists where u crosses zero.

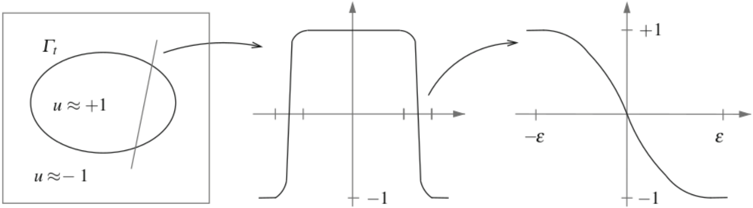


Figure: [Fig 6.4, Bartels, 15]

- Bound-preserving: $u_h(\boldsymbol{x}, t) \in [-1, 1]$
- Energy decay: $\sup_{t \in J} \mathcal{J}_\varepsilon(u_h(t)) \leq \mathcal{J}_\varepsilon(u_h^0)$
- Polynomial ε-dependence in error estimates

- Bound-preserving:
  [Li, Yang, Zhou, 20], [Du, Ju, Li, Qiao, 21],
  [Shen, Zhang, 22], [Liu, Riviere, Shen, Zhang, 24].

- Polynomial ε-dependence:
  [Feng, Prohl, 03], [Bartels, 15],
  [Chrysafinos, 19], [Akrivis, Li, 22].

Our contribution

Combine the two properties.

Let V_h be the FEM space, $V_h^+ \subset V_h$ the subset of v_h^+ s.t.:

$$\forall v_h := \sum_{i \in \mathcal{V}} V_i \varphi_i \in V_h,\ v_h^+ := \sum_{i \in \mathcal{V}} \min(1, \max(-1, V_i))\varphi_i,\ v_h^- := v_h - v_h^+.$$

Let $\alpha = \mathcal{O}(1)$. Then for all $n \in \mathcal{N}$,

$$\Big(\frac{u_h^{n,+} - u_h^{n-1,+}}{\tau} + \frac{f(u_h^{n,+})}{\varepsilon^2}, v_h\Big) + (\nabla u_h^{n,+}, \nabla v_h) + s_h(u_h^{n,-}, v_h) = 0,$$

$$s_h(u_h^{n,-}, v_h) := \alpha \sum_{i \in \mathcal{V}} \left(\frac{h_i^d}{\tau} + h_i^{d-2} + \frac{h_i^d}{\varepsilon^2}\right) U_i^{n,-} V_i, \quad \forall v_h \in V_h.$$

- Nonlinear space discretization for elliptic PDEs in [Barrenechea, Georgoulis, Pryer, Veeser, 24]
- Nonlinear system solved by a few number of Newton + Richardson-like iterations

Lemma (Well-posedness)

Assume that $\alpha = O(1)$ is sufficiently large, $\tau < \frac{\varepsilon^2}{4}$.
There exists a unique $u_h^n \in V_h$ for all $n \in \mathcal{N}$.

Proof.

Apply theory of finite-dimensional monotone operators. $\square$

Recall $\mathcal{J}_\varepsilon(v) = \frac{1}{2}\|\nabla v\|^2 + \frac{1}{\varepsilon^2}\int_D F(v)$

Lemma (Local in time)

$$u_h^{n,+} = \underset{v_h \in V_h^+}{\textit{argmin}} \left(\mathcal{J}_\varepsilon(v_h) + \frac{\tau}{2} \| \frac{v_h - u_h^{n-1,+}}{\tau} \|^2 \right)$$

Lemma (Global in time)

$$\max_{n \in \mathcal{N}} \mathcal{J}_\varepsilon(u_h^{n,+}) + C \sum_{n \in \mathcal{N}} \tau \left\| \frac{u_h^{n,+} - u_h^{n-1,+}}{\tau} \right\|^2 \le \mathcal{J}_\varepsilon(u_h^0)$$

We introduce the principal eigenvalue of the linearized eigenvalue problem from [Chen, 94]:

$$\lambda(t) := \max\left\{0, -\inf_{v\in H^1(\Omega)\setminus\{0\}} \frac{(\varepsilon^{-2}f'(u(t)), v^2) + \|\nabla v\|^2}{\|v\|^2}\right\}$$

which satisfies

$$\int_0^T \lambda(t)\mathrm{d}t \leq C + \log(\varepsilon^{-\kappa})$$

where κ represents the number of topological changes [Bartels and Müller, 11], [Bartels, Müller, Ortner, 11], [Bartels, 15]

- Split error into consistent, stability parts
- Control consistent part by a bound-preserving Ritz-projection [this thesis]
- Control stability part by linearized eigenvalue problem & generalized Grönwall inequality [Bartels, 15]

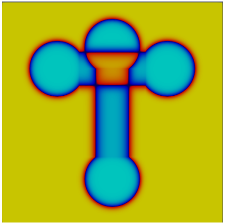
Theorem (Error estimate)

Suppose that $u \in H^1(J; H^2(\Omega)) \cap H^2(J; L^2(\Omega))$.
Let α *be sufficiently large,* $\tau, h \leq C\varepsilon^{\beta}$, *with some* $\beta > 0$.

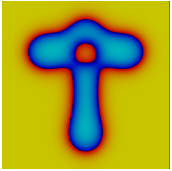
$$\max_{n\in\mathcal{N}} \|u^n - u_h^{n,+}\| \leq C\frac{1}{\varepsilon^{4\kappa+2}}(\tau + h)$$

$$\Big\{\sum_{n\in\mathcal{N}} \tau\|\nabla(u^n - u_h^{n,+})\|^2\Big\}^{\frac{1}{2}} \leq C\frac{1}{\varepsilon^{4\kappa+3}}(\tau + h)$$

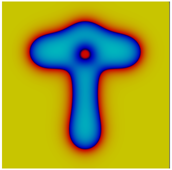
- Test from [Feng, Wu, 2005]
- $\varepsilon^2 = 2.5e-4$, $\tau = 3e-3$, $t_1 \approx 1.7e-2$, $t_2 \approx 2.5e-2$



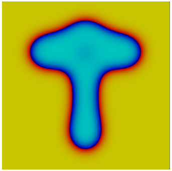
u_0



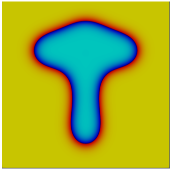
(a) operator-splitting, t_1



(b) operator-splitting, t_2



(c) our scheme, t_1



(d) our scheme, t_2

- Bound-preserving scheme

- Error bounds with polynomial ε-dependence

- Future research directions:
  - high-order schemes
  - mesh adaptivity
  - application on semi-implicit schemes

04

Linear acoustic wave equation

[Dong, Mascotto, Wang, submitted, Numer. Math.]

[Dong, Georgoulis, Mascotto, Wang, submitted, Numer. Math.]

$$\partial_{tt} u - \Delta u = f \qquad \text{in } D \times (0,T]$$
$$u(\cdot,0) \in H_0^1(D) \qquad \partial_t u(\cdot,0) \in L^2(D)$$

- hp-a priori and a posteriori analysis
  for second-order formulation are scarce
- final goal ▸ water wave
- Our contribution
  - $>$ hp-a priori analysis for fully discretized scheme
  - $>$ hp-a posteriori analysis for time semi-discretized scheme
  - $>$ hp-space-time a posteriori analysis for fully discretized scheme (submitted recently)

Thank you for your attention